Convexity and non-Euclidean Geometries

DSC DISSERTATION

Ákos G.Horváth

Department of Geometry
Institute of Mathematics
Budapest University of Technology and Economics
Budapest

January, 2017

Contents

Introduction

Overview

The dissertation contains new theorems from fourteen publications, each from the area of non-Euclidean geometries, which constitute an essential part of my research during the period of 1996 – 2016 following the defence of my candidate's degree.
Since János Bolyai, the investigation of non-Euclidean geometries has become a great tradition in Hungarian mathematical culture. This dissertation continues this tradition. We deal with problems that can be connected to non-Euclidean geometries through the bridge of convexity. These investigations are interesting for some researchers in other disciplines, e.g. programmers, physicists, engineers, geologists, and mathematicians from other areas of mathematics. We organized our dissertation to an Overview, three Chapters and an Appendix. The Overview contains a short comment on the selection of the papers included in the Thesis and a more detailed description of the results and the corresponding tools.

The structure of the dissertation. `The first chapter` contains problems from Euclidean geometry which can be solved using non-Euclidean geometric tools, or an analogous non-Euclidean problem leads to a deep result in it. As an example I mention Theorem 1.1.2 which transforms a Euclidean problem into a question in Minkowski geometry (is called by Minkowski normed space, too). If, for an n-dimensional convex body K, we have that $\mathrm{vol}(\mathrm{conv}((v+K)\cup(w+K)))$ has the same value for any touching pair of translates of K, we say that K satisfies the *translative constant volume property*. Recall that a 2-dimensional o-symmetric convex curve is a Radon curve, if, for the convex hull K of a suitable affine image of the curve, it holds that its polar K° is a rotated copy of K by $\frac{\pi}{2}$ (cf. [**117**]); the concept of Radon curves arose in connection with Birkhoff orthogonality in Minkowski normed spaces. With Zsolt Lángi we proved that for any plane convex body K the following are equivalent.

(1) K satisfies the translative constant volume property.
(2) The boundary of $\frac{1}{2}(K-K)$ is a Radon curve.
(3) K is a body of constant width in a Radon norm.

This chapter is based on three papers of the author [**12**], [**13**], [**14**] from which the paper [**13**] is a joint work with Zsolt Lángi. These results are strongly connected to three other papers of the author ([**81**], [**82**] and [**84**]).
`In the second chapter` we investigate the basic concepts of a normed space from the concept of bisector to the concept of certain important curves. A characteristic result is Theorem 2.1.7. Here we considered the topological connection between the shadow boundary of the unit ball of a Minkowski space in a given direction and the bisectors of the space corresponding to the same direction. As a good tool we introduce the concept of general parameter spheres as follows: Let K be the unit ball of the Minkowski space and x be a fixed direction of the space E^n. Denote by H_x the set of those points of the space which distance from the origin is equal to its distance from the point x. Let $\lambda_0 := \inf\{0 < t \in \mathbb{R} \mid tK \cap (tK+x) \neq \emptyset\}$ be the smallest value of t for which tK and $tK+x$ intersect. Then a *general parameter sphere of* $\mathrm{bd}K$ corresponding to the direction x and to any fixed parameter $\lambda \geq \lambda_0$ is the following set: $\gamma_\lambda(K,x) := \frac{1}{\lambda}(\mathrm{bd}(\lambda K) \cap \mathrm{bd}(\lambda K + x)) \subset \mathrm{bd}\, K$. We proved the following statement: Assume that the bisector H_x is a topological plane of E^3. Then the general parameter spheres $\gamma_\lambda(K,x)$ for $\lambda > \lambda_0$ and the shadow boundary $S(K,x)$ are topological 1-manifolds (topological circles).

For $\lambda = \lambda_0$ the parameter sphere can form a point, a segment or a convex disk of dimension 2, respectively.
This chapter contains results from seven papers [1], [2], [3], [4], [5], [6], [7] from these [4] and [6] are common works with Horst Martini, and the paper [7] is a common work with Vitor Balestro and Horst Martini. The paper [85], which is also connected to the examined problems through many ideas we omit from the dissertation because the corresponding investigation was initiated by my coauthors Zsolt Lángi and Margarita Spirova. This chapter is the backbone of the dissertation containing several tools for all other proofs, and a lot of new concepts.
`The third chapter` contains new constructions of manifold-like structures. First we introduce a common frame for Minkowski normed spaces Minkowski space-time; that is, we define a structure that contains both concepts as special cases. This concept leads to the idea of generalized Minkowski spaces which can be generalized to a model with changing shape. We call it generalized Minkowski space-time model with changing shape. In this structure the measure of the space-like component at a fixed moment depends on a norm which corresponds to the given moment of time. Since the localization in time determines the measure of lengths, we can associate to this model a shape-function. This shape-function could be either a deterministic function or a random function. Hence we get either a deterministic or a random time-space model, respectively. As Theorem 3.4.2 states, from cosmological point of view there is no essential difference between the two models. More precisely, let $\mathcal{K}_0$ be the metric space of centrally symmetric convex bodies endowed with Hausdorff metric. In Section 3.3 we define a probability measure P on it holding some important geometric properties. Let $(K_\tau\ ,\ \tau \geq 0)$ be a random function defined as an element of the Kolmogorov extension $\left(\Pi\mathcal{K}_0, \hat{P}\right)$ of the probability space $(\mathcal{K}_0, P)$. We say that the generalized space-time model endowed with the random function $\hat{K}_\tau := \sqrt[n]{\mathrm{vol}(B_E)/\mathrm{vol}(K_\tau)}K_\tau$ defines a *random time-space model*. It is clear that a deterministic time-space model is a special trajectory of the random time-space model. Theorem 3.4.2 states the following: For a trajectory $L(\tau)$ of the random time-space model, for a finite set $0 \leq \tau_1 \leq \cdots \leq \tau_s$ of moments and for some $\varepsilon > 0$ there is a deterministic time-space model defined by the (deterministic) function $K(\tau)$ for which $\sup_i\{\rho_H\left(L(\tau_i), K(\tau_i)\right)\} \leq \varepsilon$.
The chapter contains selected results from the papers [8], [9], [10], and [11].
`In the appendix` we develop the special and general relativity theory of our time space. In a mathematical dissertation the physical content of the appendix cannot be considered as a main mathematical result but it is very important to check the relevance of the conceptualization in practice. This is the reason why we add it to the dissertation.
This dissertation (due to length constraints) does not contain all the statements and examples of the mentioned papers. For further information please read the original papers in the separated literature. The description of the historical background and the precise introduction of the problem immediately precedes the result in the text. Every theorem has a reference to the original work from which it is cited. In the dissertation we also collected our examples, definitions, theorems and conjectures in an index page titled by "Index". Here we can find the number of the page where the item first appeared.

Detailed description of the content.

The first chapter. is the least homogeneous chapter, its total length is about 22 pages and contains 6 figures.
`The first section` is based on the paper [12] which is a common work with Zsolt Lángi. The problem seems to be a classical Euclidean one to determine the volume of the convex hull of two convex bodies. It has been in the focus of research since the 1950s. One of the first results in this area is due to Fáry and Rédei [55], who proved that if one of the bodies is translated on a line at a constant velocity, then the volume of their convex hull is a convex function of time. This result was reproved by Rogers and Shephard [131] in 1958, using a more general theorem about the so-called *linear parameter systems*, and for polytopes by Ahn, Brass and Shin [15]

in 2008. To generalize it we investigated the following quantities. For two convex bodies K and L in $\mathbb{R}^n$, let

$$c(K,L) = \max\{\mathrm{vol}(\mathrm{conv}(K' \cup L')) : K' \cong K, L' \cong L \text{ and } K' \cap L' \neq \emptyset\},$$

where vol denotes n-dimensional Lebesgue measure. Furthermore, if $\mathcal{S}$ is a set of isometries of $\mathbb{R}^n$, we set

$$c(K|\mathcal{S}) = \frac{1}{\mathrm{vol}(K)} \max\{\mathrm{vol}(\mathrm{conv}(K \cup K')) : K \cap K' \neq \emptyset, K' = \sigma(K) \text{ for some } \sigma \in \mathcal{S}\}.$$

We note that a quantity similar to $c(K,L)$ was defined by Rogers and Shephard [131], in which congruent copies were replaced by translates. Another related quantity is investigated in [81], where the author examines $c(K,K)$ in the special case that K is a regular simplex and the two congruent copies have the same centre.

In [131], Rogers and Shephard used linear parameter systems to show that the minimum of $c(K|\mathcal{S})$, taken over the family of convex bodies in $\mathbb{R}^n$, is its value for an n-dimensional Euclidean ball, if $\mathcal{S}$ is the set of translations or that of reflections about a point. Nevertheless, their method, approaching a Euclidean ball by suitable Steiner symmetrizations and showing that during this process the examined quantities do not increase, does not characterize the convex bodies for which the minimum is attained; they conjectured that, in both cases, the minimum is attained only for ellipsoids (cf. p. 94 of [131]). We note that the method of Rogers and Shephard [131] was used also in [110].

We treated these problems in a more general setting. For this purpose, let $c_i(K)$ be the value of $c(K|\mathcal{S})$, where $\mathcal{S}$ is the set of reflections about the i-flats of $\mathbb{R}^n$, and $i = 0, 1, \ldots, n-1$. Similarly, let $c^{tr}(K)$ and $c^{co}(K)$ be the value of $c(K|\mathcal{S})$ if $\mathcal{S}$ is the set of translations and that of all the isometries, respectively. We examined the minima of these quantities. In particular, in Theorem 1.1.1, we give another proof that the minimum of $c^{tr}(K)$, over the family of convex bodies in $\mathbb{R}^n$, is its value for Euclidean balls, and show also that the minimum is attained if, and only if, K is an ellipsoid. This verifies the conjecture in [131] for translates. In Theorem 1.1.2, we characterized the plane convex bodies for which $c^{tr}(K)$ is attained for any touching pair of translates of K, showing a connection of the problem with Radon norms. This shows that Minkowski geometric investigations can get information on Euclidean problems. In Theorems 1.1.3 and 1.1.4, we present similar results about the minima of $c_1(K)$ and $c_{n-1}(K)$, respectively. In particular, we prove that, over the family of convex bodies, $c_1(K)$ is minimal for ellipsoids, and $c_{n-1}(K)$ is minimal for Euclidean balls. The first result proves the conjecture of Rogers and Shephard for copies reflected about a point.

We used in the proof a sort of classical volume inequalities, and ad hoc observations from n-dimensional convex geometry. We had to use also some information on the orthogonality of a Minkowski normed plane to get for example the result cited in the preceding subsection.

`The second section` is based on the paper [13]. The problem of finding the maximal volume polyhedra in $\mathbb{R}^3$ with a given number of vertices and inscribed in the unit sphere, was first mentioned in [57] in 1964. A systematic investigation of this question starts with the paper [25] of Berman and Hanes in 1970, who found a necessary condition for optimal polyhedra, and determined those with $n \leq 8$ vertices. The same problem was examined in [127], where the author presented the results of a computer-aided search for optimal polyhedra with $4 \leq n \leq 30$ vertices. Nevertheless, according to our knowledge, this question, which is listed in both research problem books [31] and [39], is still open for polyhedra with $n > 8$ vertices apart from the fortunate case of $n = 12$ where the solution is the regular icosahedron. In [84] the authors investigated this problem for polytopes in arbitrary dimensions. By generalizing the methods of [25], the authors presented a necessary condition for the optimality of a polytope. The authors found the maximum volume polytopes in $\mathbb{R}^d$, inscribed in the unit sphere $\mathbb{S}^{d-1}$, with $n = d+2$ vertices; for $n = d+3$ vertices, they found the maximum volume polytope for d odd, over the family of all polytopes, and for d even, over the family of not cyclic polytopes, respectively. Observe that in this investigation spherical trigonometry plays an important role,

which is the reason why the problem is included in this section. One of the most important tools in the treatment of the 3-dimensional problem is the result of L. Fejes-Tóth on volume bounds on polyhedra inscribed in the unit sphere (formula (2) on p. 263 in [57]). For simplicial polyhedra it can be simplified into another one (see p.264 in [57]) which we call *icosahedron inequality*. The term is motivated by the fact that this inequality implies the case of $n = 12$ points when the unique solution is the icosahedron.
The aim of this section is to give similar inequalities for cases when certain (other than the number of vertices) prescribed information on the examined class of polytopes inscribed in the unit sphere need to be taken into consideration. We generalize the icosahedron inequality for simplicial bodies whose faces have given lengths of maximal edges (cf. Prop. 1.2.2, Prop. 1.2.3, Theorem 1.2.1). Our extracted formula is valid not only for convex polyhedra but also for polyhedra that area star-shaped with respect to the origin (cf. Theorem 1.2.1). As an application of the generalized inequality we prove a conjecture which states that the maximal volume polyhedron spanned by the vertices of two regular simplices with common centroid is the cube. This conjecture was raised and proved partially in [81] and inspired some other examinations on the volume of the convex hull of simplices [82]. The numerous calculations of the proof of Theorem 1.2.1 can be found in [83].
`The third section` contains a result from the paper [14]. Our observations on the volume of hyperbolic orthoscemes concerns a deficiency in the two hundred years literature. Using hyperbolic orthogonal coordinates we discovered a formula on the volume of the orthosceme by its edge lengths. Of course, our formula also contains a non-elementary integral, but it completes the collection of integrals of Lobachevsky and Bolyai to a complete triplet. (The integral of Lobachevsky uses the dihedral angles of the orthosceme and the formulas of Bolyai both the dihedral angles and the edge lengths of the orthosceme.) In this paper we described three types of coordinate systems in which the volume of a set can be given by an appropriate integral. These coordinate systems are based on a parasphere, the hyperbolic orthogonal coordinate system and the spherical coordinate system, respectively. Using these we determined the volume form with respect to these coordinate systems and also with respect to the half-space and projective model. To determine these formulas we need some information on hyperbolic trigonometry and also some well-known analytic and synthetic results from hyperbolic geometry. The formulas can be get from each other by (non-trivial) integral transforms and so we had to give only the first one by a synthetic native reasoning. The dissertation contains only those steps which are needed to the deduction of the required formula on orthoscheme: Let denote by a, b and c those edges (and their lengths) of the orthosceme for which b is orthogonal to a and c is orthogonal to a and b, respectively. Then for the volume v of the orthosceme we have:

$$v = \frac{1}{4}\int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda.$$

The second chapter. presents the basis of the dissertation. In recent times, the geometry of finite dimensional, real Banach spaces; see [140] became again an important research field. Strongly related to Banach space theory, it is permanently enriched by new results in applied disciplines. The most examined concepts of it naturally connect to physics, functional analysis, and non-Euclidean geometries. Our eight publications studied the geometric structure of a Minkowski normed space, especially the problems of bisectors, conics, roulettes, isometries and polarities. The total length of this part of the dissertation is about 50 pages with 26 figures.
`The first section` is based on four papers from which one ([4]) has a co-author, Horst Martini. The remaining three articles ([1, 2, 3]) contain the first systematic investigations of the bisectors in higher-dimensional spaces. On a Minkowski normed plane the concept of bisector was intensively studied from the beginning (see the survey [115]), however, in higher-dimensional spaces there are only sporadic results. The reason is the complicated topology of high dimensional bisectors. We consider the following questions: What is the connection between the

topology of the bisector and the unit sphere of the Minkowski space? What is the connection between the bisector and the shadow boundary in a given direction of the space? How can we represents the bisector "well" in the unit ball of the space? We examined in [1] the boundary of the unit ball of the norm and present two theorems similar to the characterization of the Euclidean norm investigated by H.Mann, A.C.Woods and P.M.Gruber in [111], [147], [74], [75] and [76], respectively. H.Mann proved that a Minkowski normed space is Euclidean (so its unit ball is an ellipsoid) if and only if all *Leibnizian halfspaces* (containing those points of the space which are closer to the origin than to another point x) are convex. A.C.Woods proved the analogous statement for such a distance function whose unit ball is bounded but is not necessarily centrally symmetric or convex. P.M Gruber extended the theorem for distance functions whose unit ball is a ray set. P.M. Gruber generalized Woods's theorem in another way, too. He showed (see Satz.5 in [74]) that a bounded distance function gives a Euclidean norm if and only if there is a subset T of the $(n-1)$-dimensional unit sphere whose relative interior (with respect to the sphere) is not empty, having the property that for each pair of points {0,x}, where $x \in T$, the corresponding Leibnizian halfspace is convex. From the convexity of the Leibnizian halfspaces follows that the collection of all points of the space whose distances from two distinct points are equal are hyperplanes. We call such a set the *bisector* of the considered points. Thus from Mann's theorem follows a theorem stated first explicitly by M.M.Day in [42]: All of the bisectors, with respect to the Minkowski norm defined by the body K, are hyperplanes if and only if K is an ellipsoid. In this part my main result is the fact that the bisectors of a strictly convex Minkowski normed space are always homeomorphic to a hyperplane but the reverse direction of this statements is not true. We give an example for a Minkowski space in which the bisectors are homeomorphic hyperplanes but the unit ball is not strictly convex. The mathematical tools of the proofs are from convex geometry, and from basic combinatorial topology combined with Euclidean geometric observations.
To answer the second question we formulated a conjecture (Conjecture 2.1.2) which states that the bisectors are topological $(n-1)$-dimensional hyperplanes if and only if the corresponding shadow boundaries are $(n-2)$-dimensional topological spheres. In [2] and (in the third subsection of this section) we prove this conjecture in the three-dimensional case. We examined also the topological properties of the shadow boundary, and defined the so-called *general parameter spheres* for $n \geq 3$, as a tool for a prospective proof of our conjecture. The main mathematical tool of this section is the Schoenflies-Swingle theorem on the arc-wise accessibility of a curve from a domain. This theorem holds only in a two-dimensional manifold and there is no analogous characterization in higher spaces so the method of the proof cannot be extracted to higher dimensions. In [3] (and in Subsection 2.1.4) we examined the conjecture in higher than three-dimensional cases. It requires a deeper investigation of the topological properties of the general parameter spheres. We proved that the general parameter spheres are not an absolute neighborhood retract in general, but still are compact metric spaces, containing $(n-2)$-dimensional closed, connected subsets separating the boundary of K. Thus we investigated the manifold case and proved that the general parameter spheres and the corresponding shadow boundary are homeomorphic to the $(n-2)$-dimensional sphere. Furthermore, if it is an $(n-1)$-dimensional manifold with boundary then it is homeomorphic to the cylinder $S^{(n-2)} \times [0,1]$. The proof is based on geometric topology, on the so-called *cell-like approximation theorem for manifolds*. We also proved on the connection of the shadow boundary $S(K,x)$ and the general parameter spheres the following:

- $S(K,x)$ is an $(n-2)$-dimensional manifold if all of the non-degenerated general parameter spheres $\gamma_\lambda(K,x)$ with $\lambda > \lambda_0$ are $(n-2)$-dimensional manifolds, and conversely, if $S(K,x)$ is an $(n-2)$-dimensional manifold then all of the general parameter spheres are ANRs.
- $S(K,x)$ is an $(n-1)$-dimensional manifold with boundary if and only if there is a λ for which the general parameter sphere $\gamma_\lambda(K,x)$ is an $(n-1)$-dimensional manifold with boundary.

Combining these theorems and using a topological theorem of M. Brown we get the proof of the first direction of the conjecture.
By Horst Martini we continued the investigation of bisectors in a further point of view in [4]. Martini and Wo in [118] introduced and investigated the radial projection of the bisector. In our common paper with H. Martini we introduced the bounded representation of bisectors, which yields a useful combination of the notions of bisector, shadow boundary, and radial projection. We proved that the topological properties of the radial projection (in higher dimensions) do not determine the topological properties of the bisector. More precisely, the manifold property of the bisector does not imply the manifold property of the radial projection. The situation is different with respect to the bounded representation of the bisector. Namely, if one of them is a manifold, then the other one is also. More precisely, if the bisector is a manifold of dimension $(n-1)$, then its bounded representation is homeomorphic to a closed $(n-1)$-dimensional ball (i.e., it is a cell of dimension $(n-1)$). And conversely, if the bounded representation is a cell, then the closed bisector is also.
`The second section` is based on the new results of the paper [5]. It contains investigations on two types of the important transformations of a Minkowski normed space. Especially we considered "adjoint abelian" and isometric transformations of a Minkowski space. Stampfli in [136] has defined a bounded linear operator A to be *adjoint abelian* if and only if there is a duality map φ such that $A^*\varphi = \varphi A$. So evidently, A is adjoint abelian if and only if $A = A^T$, thus the adjoint abelian operators are in some sense "self-adjoint" ones. Lángi in [101] introduced the concept of the *Lipschitz property* of a semi inner product and investigated the diagonalizable operators of a Minkowski geometry $\{V, \|\cdot\|\}$. As a corollary of his main result we have that in a totally non-Euclidean Minkowski n-space every diagonalizable adjoint abelian operator is a scalar multiple of an isometry. First we described the structure of an adjoint abelian operator in Theorem 2.2.3 then in Theorem 2.2.4 we proved that in an l_p space every adjoint abelian operator is diagonalizable.
On isometries we have also two theorems. Theorem 2.2.8 describes the structure of an isometry and Theorem 2.2.10 characterizes the group of isometries as follows: If the unit ball $\mathcal{B}$ of $(V, \|\cdot\|)$ has no intersection with a two-plane which is an ellipse, then the group $\mathcal{I}(3)$ of isometries of $(V, \|\cdot\|)$ is isomorphic to the semi-direct product of the translation group $\mathcal{T}(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.
`The third section` contain results from two further papers which are important in the setting up of a complete image on our works in Minkowski geometry. These are common papers with H. Martini ([6]) and with V. Balestro and H.Martini ([7]), respectively. Due to the limitation on the length of the dissertation in this section we omit the proofs which use convex geometry, linear algebra and classical differential geometry. The paper [6] on conics contains the possible metric definitions of conics and the basic properties of the curves defined in this way. The paper [7] dealing with a possible definition of roulettes is based on a new concept of rotations. Though our rotations are not isometries implying that the motion defined by them is not a rigid one, there is a complete building up of the kinematics in a Minkowski plane. In this theory the two Euler-Savary equations are valid.

The third chapter. deals with the problem of conceptualization. The one hundred old concept of "Minkowski space" is a central topic of the scientific community. Note that the phrase "Minkowski space" do not distinguish between two theories: the theory of normed linear spaces and the theory of linear spaces with indefinite metric. For finite dimensions both are called Minkowski spaces in the literature. It is interesting that these essentially distinct theories of mathematics have similar axiomatic foundations. The axiomatic examination of the theory of linear spaces with indefinite metric comes from H. Minkowski [123] and the similar system of axioms of normed linear spaces was introduced by Lumer in [108]. The first concept widely used in physics: this is the mathematical structure of relativity theory and thus there is no doubt about its importance. (The popularity of linear spaces with indefinite metric is undiminished since Minkowski's lecture "Time and Space".) The usability of the second one is based

on the fact that modern functional analysis works in general normed spaces, and the Lumer-Giles theory of semi inner product gives a possibility to handling it by methods used originally in Hilbert spaces. Of course, in both of these spaces there are a lot of problems that can be formulated or solved in the language of geometry. The results of this chapter can be found in four publications of the author [**8**, **9**, **10**, **11**].

The two publications [**8**],[**9**] are about the new concept of generalized space-time model. The fourth paper [**11**] extend this concept to a concept of generalized Minkowski space with changing shape, distinguishing to each other the random and deterministic possibilities. For this purpose we had to define a probability space on the metric space of centrally symmetric convex compact bodies. The third paper [**10**] contains a construction in this direction. In this introductory section I would not like to present a more detailed description of the content of this chapter, I remark only two things. First of all, the aim of this part of the dissertation is concept rendering, which means that the purpose of the theorems is the verification of conceptualization. Secondly, for this natural reason the used mathematical tools are very dispersed, we had to apply results from linear algebra, functional analysis, convex geometry, probability theory and also classical and modern differential geometry. The sum of the lengths of the four papers is 103 pages, from this the dissertation contains a 50 page long review. As an application of this theory we add an Appendix to the dissertation. It contains the description of the relativity theory in our structure from the special relativity to the Einstein equation holding in a time-space manifold.

dc_1387_17

CHAPTER 1

Problems on convexity and volumes in connection with non-Euclidean geometries

1.1. On the convex hull of two convex bodies (common work with Zs. Lángi)

We investigate the following quantities.

DEFINITION 1.1.1. [12] *For two convex bodies K and L in $\mathbb{R}^n$, let*

$$c(K,L) = \max\left\{\mathrm{vol}(\mathrm{conv}(K' \cup L')) : K' \cong K, L' \cong L \text{ and } K' \cap L' \neq \emptyset\right\},$$

where vol *denotes n-dimensional Lebesgue measure. Furthermore, if $\mathcal{S}$ is a set of isometries of $\mathbb{R}^n$, we set*

$$c(K|\mathcal{S}) = \frac{1}{\mathrm{vol}(K)} \max\left\{\mathrm{vol}(\mathrm{conv}(K \cup K')) : K \cap K' \neq \emptyset, K' = \sigma(K) \text{ for some } \sigma \in \mathcal{S}\right\}.$$

We note that a quantity similar to $c(K,L)$ was defined by Rogers and Shephard [131], in which congruent copies were replaced by translates. Another related quantity is investigated in [81], where the author examines $c(K,K)$ in the special case that K is a regular simplex and the two congruent copies have the same centre.

In [131], Rogers and Shephard used linear parameter systems to show that the minimum of $c(K|\mathcal{S})$, taken over the family of convex bodies in $\mathbb{R}^n$, is its value for an n-dimensional Euclidean ball, if $\mathcal{S}$ is the set of translations or that of reflections about a point. Nevertheless, their method, approaching a Euclidean ball by suitable Steiner symmetrizations and showing that during this process the examined quantities do not increase, does not characterize the convex bodies for which the minimum is attained; they conjectured that, in both cases, the minimum is attained only for ellipsoids (cf. p. 94 of [131]). We note that the method of Rogers and Shephard [131] was used also in [110].

We treat these problems in a more general setting. For this purpose, let $c_i(K)$ be the value of $c(K|\mathcal{S})$, where $\mathcal{S}$ is the set of reflections about the i-flats of $\mathbb{R}^n$, and $i = 0, 1, \ldots, n-1$. Similarly, let $c^{tr}(K)$ and $c^{co}(K)$ be the value of $c(K|\mathcal{S})$ if $\mathcal{S}$ is the set of translations and that of all the isometries, respectively.

During the investigation, $\mathcal{K}_n$ denotes the family of n-dimensional convex bodies. Let $\mathcal{B}^n$ be the n-dimensional unit ball with the origin o of $\mathbb{R}^n$ as its centre, and set $\mathbb{S}^{n-1} = \mathrm{bd}\,\mathcal{B}^n$ and $v_n = \mathrm{vol}(\mathcal{B}_n)$. Finally, we denote 2- and $(n-1)$-dimensional Lebesgue measure by area and vol_{n-1}, respectively. For any $K \in \mathcal{K}_n$ and $u \in \mathbb{S}^{n-1}$, $K|u^\perp$ denotes the orthogonal projection of K into the hyperplane passing through the origin o and perpendicular to u. The *polar* of a convex body K is denoted by K°.

THEOREM 1.1.1. [12] *For any $K \in \mathcal{K}_n$ with $n \geq 2$, we have $c^{tr}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$ with equality if, and only if, K is an ellipsoid.*

PROOF. By compactness arguments, the minimum of $c^{tr}(K)$ is attained for some convex body K, and since for ellipsoids it is equal to $1 + \frac{2v_{n-1}}{v_n}$, it suffices to show that if $c^{tr}(K)$ is minimal for K, then K is an ellipsoid.

Let $K \in \mathcal{K}_n$ be a convex body such that $c^{tr}(K)$ is minimal. Then $c^{tr}(K) \leq 1 + \frac{2v_{n-1}}{v_n}$. For any $u \in \mathbb{S}^{n-1}$, let $d_K(u)$ denote the length of a maximal chord parallel to u. Observe that for any

such u, K and $d_K(u)u+K$ touch each other and

$$\frac{\mathrm{vol}(\mathrm{conv}(K\cup(d_K(u)u+K)))}{\mathrm{vol}(K)}=1+\frac{d_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp)}{\mathrm{vol}(K)}. \tag{1}$$

Clearly, $c^{tr}(K)$ is the maximum of this quantity over $u\in\mathbb{S}^{n-1}$.
It is known that for any K and u, $d_K(u)=d_{\frac{1}{2}(K-K)}(u)$ and the same holds also for the width function of K. Theorem 3.3.5 of [**63**] states that if K and K' have the same width function, then they have the same brightness function, defined as $u\mapsto\mathrm{vol}_{n-1}(K|u^\perp)$, as well. Thus, we have that for any $u\in\mathbb{S}^{n-1}$, $d_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp)=d_{\frac{1}{2}(K-K)}(u)\,\mathrm{vol}_{n-1}\left(\frac{1}{2}(K-K)|u^\perp\right)$. On the other hand, the Brunn-Minkowski Inequality yields that $\mathrm{vol}(K)\le\mathrm{vol}\left(\frac{1}{2}(K-K)\right)$, with equality if, and only if, K is centrally symmetric. Substituting these inequalities into (1), we obtain that $c^{tr}(K)\ge c^{tr}\left(\frac{1}{2}(K-K)\right)$, with equality if, and only if, K is centrally symmetric. Hence, in the following we may assume that K is o-symmetric.
Let $u\mapsto r_K(u)=\frac{d_K(u)}{2}$ be the radial function of K. From (1) and the inequality $c^{tr}(K)\le 1+\frac{2v_{n-1}}{v_n}$, we obtain that for any $u\in\mathbb{S}^{n-1}$

$$\frac{v_{n-1}\,\mathrm{vol}(K)}{v_n\,\mathrm{vol}_{n-1}(K|u^\perp)}\ge r_K(u). \tag{2}$$

Applying this for the polar form of the volume of K, we obtain

$$\mathrm{vol}(K)=\frac{1}{n}\int\limits_{\mathbb{S}^{n-1}}(r_K(u))^n\,\mathrm{d}\,u\le\frac{1}{n}\frac{v_{n-1}^n}{v_n^n}\,(\mathrm{vol}(K))^n\int\limits_{\mathbb{S}^{n-1}}\frac{1}{(\mathrm{vol}_{n-1}(K|u^\perp))^n}\,\mathrm{d}\,u,$$

which yields

$$\frac{v_n^n n}{v_{n-1}^n\,(\mathrm{vol}(K))^{n-1}}\le\int\limits_{\mathbb{S}^{n-1}}\frac{1}{(\mathrm{vol}_{n-1}(K|u^\perp))^n}\,\mathrm{d}\,u \tag{3}$$

On the other hand, combining Cauchy's surface area formula with Petty's projection inequality, we obtain that for every $p\ge -n$,

$$v_n^{1/n}\,(\mathrm{vol}(K))^{\frac{n-1}{n}}\le v_n\left(\frac{1}{nv_n}\int\limits_{S^{n-1}}\left(\frac{\mathrm{vol}_{n-1}(K|u^\perp)}{v_{n-1}}\right)^p\mathrm{d}\,u\right)^{\frac{1}{p}},$$

with equality only for Euclidean balls if $p>-n$, and for ellipsoids if $p=-n$ (cf. e.g. Theorems 9.3.1 and 9.3.2 in [**63**]).
This inequality, with $p=-n$ and after some algebraic transformations, implies that

$$\int\limits_{\mathbb{S}^{n-1}}\frac{1}{(\mathrm{vol}_{n-1}(K|u^\perp))^n}\,\mathrm{d}\,u\le\frac{v_n^n n}{v_{n-1}^n\,(\mathrm{vol}(K))^{n-1}} \tag{4}$$

with equality if, and only if K, is an ellipsoid. Combining (3) and (4), we can immediately see that if $c^{tr}(K)$ is minimal, then K is an ellipsoid, and in this case $c^{tr}(K)=1+\frac{2v_{n-1}}{v_n}$. □

We remark that a theorem related to Theorem 1.1.1 can be found in [**112**]. More specifically, Theorem 11 of [**112**] states that for any convex body $K\in\mathcal{K}_n$, there is a direction $u\in\mathbb{S}^{n-1}$ such that, using the notations of Theorem 1.1.1, $d_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp)\ge\frac{2v_{n-1}}{v_n}$, and if for any direction u the two sides are equal, then K is an ellipsoid.
If, for a convex body $K\in\mathcal{K}_n$, we have that $\mathrm{vol}(\mathrm{conv}((v+K)\cup(w+K)))$ has the same value for any touching pair of translates, let us say that K satisfies the *translative constant volume property*. In this section we will characterize the plane convex bodies with this property. Before doing this, we recall that a 2-dimensional o-symmetric convex curve is a Radon curve, if, for the convex hull K of a suitable affine image of the curve, it holds that K° is a rotated copy of K by

$\frac{\pi}{2}$ (cf. [117]). We note that the concept of Radon curve arose in connect with the examination of the Birkhoff orthogonality in Minkowski normed spaces.

THEOREM 1.1.2. [12] *For any plane convex body $K \in \mathcal{K}_2$ the following are equivalent.*

(1) *K satisfies the translative constant volume property.*
(2) *The boundary of $\frac{1}{2}(K-K)$ is a Radon curve.*
(3) *K is a body of constant width in a Radon norm.*

PROOF. Clearly, (2) and (3) are equivalent, and thus, we need only show that (1) and (2) are. Let $K \in \mathcal{K}_2$. For any $u \neq o$, let $d_K(u)$ and $w_K(u)$ denote the length of a maximal chord and the width of K in the direction of u. Then, using the notation $u = w - v$, for any touching pair of translates, we have

$$\operatorname{area}(\operatorname{conv}((v+K) \cup (w+K))) = \operatorname{area}(K) + d_K(u) w_K(u^{\perp}),$$

where $u^{\perp}$ is perpendicular to u.
Since for any direction u, we have $d_K(u) = d_{\frac{1}{2}(K-K)}(u)$ and $w_K(u) = w_{\frac{1}{2}(K-K)}(u)$, K satisfies the translative constant volume property if, and only if, its central symmetral does. Thus, we may assume that K is o-symmetric. Now let $x \in \operatorname{bd} K$. Then the boundary of $\operatorname{conv}(K \cup (2x+K))$ consists of an arc of $\operatorname{bd} K$, its reflection about x, and two parallel segments, each contained in one of the two common supporting lines of K and $2x+K$, which are parallel to x. For some point y on one of these two segments, set $A_K(x) = \operatorname{area}\operatorname{conv}\{o, x, y\}$ (cf. Figure 1.1). Clearly, $A_K(x)$ is independent of the choice of y. Then we have for every $x \in \operatorname{bd} K$, that $d_K(x) w_K(x^{\perp}) = 8 A_K(x)$.

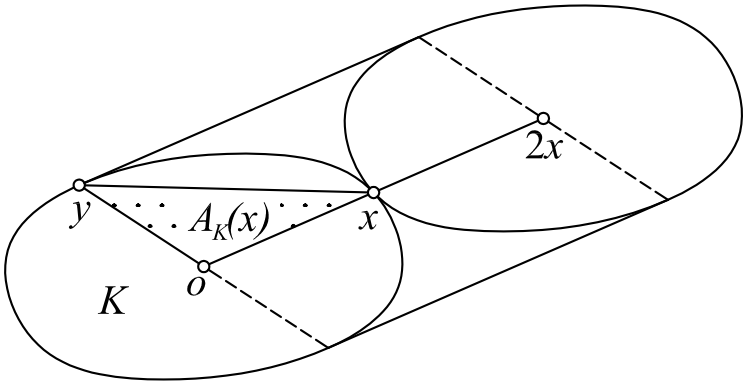

FIGURE 1.1. An illustration for the proof of Theorem 1.1.2

Assume that $A_K(x)$ is independent of K. We need to show that in this case $\operatorname{bd} K$ is a Radon curve. It is known (cf. [117]), that $\operatorname{bd} K$ is a Radon curve if, and only if, in the norm of K, *Birkhoff-orthogonality* is a symmetric relation. Recall that in a normed plane with unit ball K, a vector x is called *Birkhoff-orthogonal* to a vector y, denoted by $x \perp_B y$, if x is parallel to a line supporting $\|y\| \operatorname{bd} K$ at y (cf. [17]).
Observe that for any $x, y \in \operatorname{bd} K$, $x \perp_B y$ if, and only if, $A_K(x) = \operatorname{area}(\operatorname{conv}\{o, x, y\})$, or in other words, if, $\operatorname{area}(\operatorname{conv}\{o, x, y\})$ is maximal over $y \in K$. Clearly, it suffices to prove the symmetry of Birkhoff orthogonality for $x, y \in \operatorname{bd} K$. Consider a sequence $x \perp_B y \perp_B z$ for some $x, y, z \in \operatorname{bd} K$. Then we have $A_K(x) = \operatorname{area}\operatorname{conv}\{o, x, y\}$ and $A_K(y) = \operatorname{area}(\operatorname{conv}\{o, y, z\})$. By the maximality of $\operatorname{area}(\operatorname{conv}\{o, y, z\})$, we have $A_K(x) \leq A_K(y)$ with equality if, and only if, $y \perp_B x$. This readily implies that Birkhoff orthogonality is symmetric, and thus, that $\operatorname{bd} K$ is a Radon curve. The opposite direction follows from the definition of Radon curves and polar sets. □

THEOREM 1.1.3. [12] *For any $K \in \mathcal{K}_n$ with $n \geq 2$, $c_1(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, with equality if, and only if, K is an ellipsoid.*

PROOF. If K is centrally symmetric, then $c_1(K) = c^{tr}(K)$, and we can apply Theorem 1.1.1. Consider the case that K is not centrally symmetric. Let $\sigma : \mathcal{K}_n \to \mathcal{K}_n$ be a Steiner symmetrization about any hyerplane, and observe that $\sigma(-K) = -\sigma(K)$. Thus, Lemma 2 of [**131**] yields that $c_1(K) \geq c_1(\sigma(K))$. On the other hand, Lemma 10 of [**112**] states that, for any not centrally symmetric convex body, there is an orthonormal basis such that subsequent Steiner symmetrizations, through hyperplanes perpendicular to its vectors, yields a centrally symmetric convex body, different from ellipsoids. Combining these statements, we obtain that there is an o-symmetric convex body $K' \in \mathcal{K}_n$ that is not an ellipsoid and satisfies $c_1(K) \geq c_1(K')$. Hence, the assertion follows immediately from Theorem 1.1.1. □

Our next result shows an inequality for $c_{n-1}(K)$.

THEOREM 1.1.4. [**12**] *For any $K \in \mathcal{K}_n$ with $n \geq 2$, $c_{n-1}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, with equality if, and only if, K is a Euclidean ball.*

PROOF. For a hyperplane $\sigma \subset \mathbb{R}^n$, let K_σ denote the reflected copy of K about σ. Furthermore, if σ is a supporting hyperplane of K, let $K_{-\sigma}$ be the reflected copy of K about the other supporting hyperplane of K parallel to σ. Clearly,

$$c_{n-1}(K) = \frac{1}{\mathrm{vol}(K)} \max\{\mathrm{vol}(\mathrm{conv}(K \cup K_\sigma)) : \sigma \text{ is a supporting hyperplane of } K\}.$$

For any direction $u \in \mathbb{S}^{n-1}$, let $H_K(u)$ be the right cylinder circumscribed about K and with generators parallel to u. Observe that for any $u \in \mathbb{S}^{n-1}$ and supporting hyperplane σ perpendicular to u, we have

$$\mathrm{vol}(\mathrm{conv}(K \cup K_\sigma)) + \mathrm{vol}(\mathrm{conv}(K \cup K_{-\sigma}) = 2\,\mathrm{vol}(K) + 2\,\mathrm{vol}(H_K(u)) =$$
$$= 2\,\mathrm{vol}(K) + 2w_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp).$$

Thus, for any $K \in \mathcal{K}_n$,

$$c_{n-1}(K) \geq 1 + \frac{\max\{w_K(u)\,\mathrm{vol}_{n-1}(K|u^\perp) : u \in \mathbb{S}^{n-1}\}}{\mathrm{vol}(K)}. \tag{5}$$

Similarly like in the proof of Theorem 1.1.1, we can observe that the width and the brightness functions of K and its central symmetrals are equal, and thus, the numerator of the fraction on the right-hand side of (5) is the same for K and $\frac{1}{2}(K - K)$. On the other hand, the Brunn-Minkowski Inequality implies that $\mathrm{vol}(K) \leq \mathrm{vol}\left(\frac{1}{2}(K - K)\right)$, with equality if, and only if, K is centrally symmetric. Hence any minimizer of $c_{n-1}(K)$ is centrally symmetric.
Assume that K is o-symmetric, and let $d_K(u)$ denote the length of a longest chord of K parallel to $u \in \mathbb{S}^{n-1}$. Observe that for any $u \in \mathbb{S}^{n-1}$, $d_K(u) \leq w_K(u)$, and thus for any convex body K,

$$c_{n-1}(K) \geq c^{tr}(K).$$

This readily implies that $c^{n-1}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, and if here there is equality for some $K \in \mathcal{K}_n$, then K is an ellipsoid. On the other hand, in case of equality, for any $u \in \mathbb{S}^{n-1}$ we have $d_K(u) = w_K(u)$, which yields that K is a Euclidean ball. This finishes the proof of the theorem. □

In connection with the above results we had some remarks and conjecture. Some of them I quote here showing that in this theme there are a lot of problem for further interesting research.

CONJECTURE 1.1.1. *Let $n \geq 2$ and $0 < i < n-1$. Prove that, for any $K \in \mathcal{K}_n$, $c_i(K) \geq 1 + \frac{2v_{n-1}}{v_n}$. Is it true that equality holds only for Euclidean balls?*

The maximal values of $c^{tr}(K)$ and $c_0(K)$, for $K \in \mathcal{K}_n$, and the convex bodies for which these values are attained, are determined in [**131**]. Using a suitable simplex as K, it is easy to see that the set $\{c_i(K) : K \in \mathcal{K}_n\}$ is not bounded from above for $i = 1, \ldots, n-1$. This readily yields the same statement for $c^{co}(K)$ as well. On the other hand, from Theorem 1.1.4 we obtain the following.

REMARK 1.1.1. *For any $K \in \mathcal{K}_n$ with $n \geq 2$, we have $c^{co}(K) \geq 1 + \frac{2v_{n-1}}{v_n}$, with equality if, and only if, K is a Euclidean ball.*

In Theorem 1.1.2, we proved that in the plane, a convex body satisfies the translative equal volume property if, and only if, it is of constant width in a Radon plane. It is known (cf. [**17**] or [**117**]) that for $n \geq 3$, if every planar section of a normed space is Radon, then the space is Euclidean; that is, its unit ball is an ellipsoid. We conjecture the following.

CONJECTURE 1.1.2. *Let $n \geq 3$. If some $K \in \mathcal{K}_n$ satisfies the translative equal volume property, then K is a convex body of constant width in a Euclidean space.*

Furthermore, we remark that the proof of Theorem 1.1.2 can be extended, using the Blaschke-Santaló inequality, to prove Theorems 1.1.1 and 1.1.3 in the plane. Similarly, Theorem 1.1.4 can be proven by a modification of the proof of Theorem 1.1.1, in which we estimate the volume of the polar body using the width function of the original one, and apply the Blaschke-Santaló inequality.
Like in [**131**], Theorems 1.1.1 and 1.1.4 yield information about circumscribed cylinders. Note that the second corollary is a strenghtened version of Theorem 5 in [**131**].

COROLLARY 1.1.1. *For any convex body $K \in \mathcal{K}_n$, there is a direction $u \in \mathbb{S}^{n-1}$ such that the right cylinder $H_K(u)$, circumscribed about K and with generators parallel to u has volume*

$$\text{vol}(H_K(u)) \geq \left(1 + \frac{2v_{n-1}}{v_n}\right) \text{vol}(K). \tag{6}$$

Furthermore, if K is not a Euclidean ball, then the inequality sign in (6) is a strict inequality.

COROLLARY 1.1.2. *For any convex body $K \in \mathcal{K}_n$, there is a direction $u \in \mathbb{S}^{n-1}$ such that any cylinder $H_K(u)$, circumscribed about K and with generators parallel to u, has volume*

$$\text{vol}(H_K(u)) \geq \left(1 + \frac{2v_{n-1}}{v_n}\right) \text{vol}(K). \tag{7}$$

Furthermore, if K is not an ellipsoid, then the inequality sign in (7) is a strict inequality.

In the paper [**12**] we also introduced variants of these quantities for convex m-gons in $\mathbb{R}^2$, and for small values of m, characterize the polygons for which these quantities are minimal. It has been collected some additional remarks and questions, too.

1.2. On the volume of the convex hull of points inscribed in the unit sphere

We generalize here partially an important inequality of László Fejes-Tóth published in [**57**]. Let $a(P)$ be the area of a convex p-gon P lying in the unit sphere, $\tau(P)$ the (spherical) area of the central projection of P upon the unit sphere, and $v(P)$ the volume of the pyramid of base P and apex O which is the centre of the unit sphere. Let denote $U(\tau(P), p)$ the maximum of $v(P)$ for a given pair of values p and $\tau(P)$.

PROPOSITION 1.2.1 ([**57**]). *With the above notation we have the following statements.*

(1) For given values of p and τ the volume v attains its maximum $U(\tau, p)$ if t is a regular p-gon.

(2) For general $p \geq 3$ we have

$$U(\tau, p) = \frac{p}{3} \cos^2 \frac{\pi}{p} \tan \frac{2\pi - \tau}{2p} \left(1 - \cot^2 \frac{\pi}{p} \tan^2 \frac{2\pi - \tau}{2p}\right), \tag{8}$$

implying that

$$U(\tau, 3) = \frac{1}{4} \tan \frac{2\pi - \tau}{6} \left(1 - \frac{1}{3} \tan^2 \frac{2\pi - \tau}{6}\right), \tag{9}$$

(3) The function $U(\tau, p)$ is concave on the domain determined by the inequalities $0 < \tau \leq \pi$, $p \geq 3$.

(4) *If V denotes the volume, R the circumradius of a convex polyhedron having f faces, v vertices and e edges, then*

$$V \leq \frac{2e}{3}\cos^2\frac{\pi f}{2e}\cot\frac{\pi v}{2e}\left(1-\cot^2\frac{\pi f}{2e}\cot^2\frac{\pi v}{2e}\right)R^3. \tag{10}$$

Equality holds only for regular polyhedra.

A polyhedron with a given number n of vertices is always the limiting figure of a simplicial polyhedron with n vertices, hence, introducing the notation $\omega_n = (n\pi)/6(n-2)$ we have the following inequality

$$V \leq \frac{1}{6}(n-2)\cot\omega_n(3-\cot^2\omega_n)R^3. \tag{11}$$

Equality holds in the above inequality only for the regular tetrahedron, octahedron and icosahedron ($n = 4, 6, 12$).

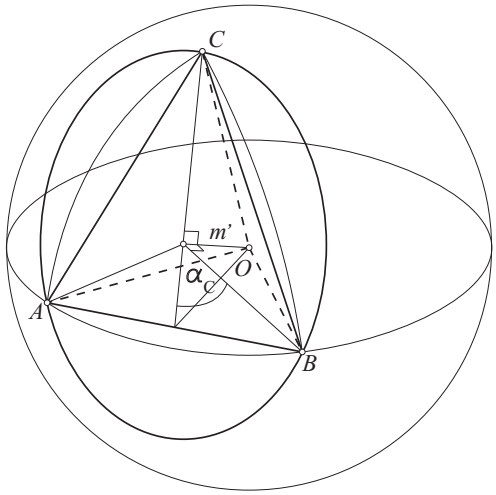

FIGURE 1.2. Facial, rectilineal and spherical simplices, respectively.

If A, B, C are three points on the unit sphere we can take two triangles with these vertices, one of the corresponding spherical triangle and the second one the rectilineal triangle with these vertices, respectively. Both of them are denoted by ABC. The angles of the rectilineal triangle are the halves of the angles between those radii of the circumscribed circle which connect the center K of the rectilineal triangle ABC to the vertices A, B, C. Since K is also the foot of the altitude of the tetrahedron with base ABC and apex O, hence the angles α_A, α_B and α_C of the rectilineal triangle ABC, play an important role in our investigations, we refer to them as the *central angle* of the spherical edges BC, AC and AB, respectively. We call the tetrahedron $ABCO$ the *facial tetrahedron* with base ABC and apex O.

PROPOSITION 1.2.2. [13] *Let ABC be a triangle inscribed in the unit sphere. Then there is an isosceles triangle $A'B'C'$ inscribed in the unit sphere with the following properties:*

- *the greatest central angles and also the spherical areas of the two triangles are equal to each other, respectively;*
- *the volume of the facial tetrahedron with base $A'B'C'$ is greater than or equal to the volume of the facial tetrahedron with base ABC.*

PROOF. Assume first that the triangle ABC contains the centre K of its circumscribed circle. Let us denote by K' the central projection of K onto the unit sphere. The angles $2\alpha_A$ and β_A are the spherical angles of the triangle $K'BC$ at K' and B (or C), respectively. Then the area of the triangle KBC is equal to $a(KBC) = \Delta(\alpha_A, \beta_A) = \frac{1}{2}\sin 2\alpha_A \sin^2 K'OB\angle = \frac{1}{2}\sin 2\alpha_A\,(1-\cot^2\alpha_A\cot^2\beta_A)$. On the domain $0 \leq \alpha \leq \frac{\pi}{2}$, $\quad 0 \leq \beta \leq \frac{\pi}{2}$, $\quad \alpha+\beta \geq \frac{\pi}{2}$ it is a concave function of two variables (see p.267 in [57]). Hence

$$a(ABC) = \Delta(\alpha_A,\beta_A)+\Delta(\alpha_B,\beta_B)+\Delta(\alpha_C,\beta_C) \leq 2\Delta\left(\frac{\alpha_A+\alpha_B}{2},\frac{\beta_A+\beta_B}{2}\right)+\Delta(\alpha_C,\beta_C) = a(A'B'C'),$$

where the value on the right hand side of the inequality above is the area of the isosceles triangle $A'B'C'$. (We note that the central projections upon the sphere of the two triangles have the same spherical excess $a(ABC) = a(A'B'C') = 2(\beta_A+\beta_B+\beta_C)-\pi$.)

Compare now the altitudes m and m' of the pyramids based on the two triangles, respectively. The spherical area of the first triangle is

$$\tau = 2(\beta_A+\beta_B+\beta_C)-\pi = 2\pi+(2(\beta_A+\beta_B+\beta_C)-3\pi) =$$

$$= 2\pi+2\left(\tan^{-1}\left(\tan\left(\beta_A-\frac{\pi}{2}\right)\right)++\tan^{-1}\left(\tan\left(\beta_B-\frac{\pi}{2}\right)\right)+\tan^{-1}\left(\tan\left(\beta_C-\frac{\pi}{2}\right)\right)\right) =$$

$$= 2\pi - 2\left(\tan^{-1}(m\tan\alpha_A) + \tan^{-1}(m\tan\alpha_B) + \tan^{-1}(m\tan\alpha_C)\right).$$

Since we do not exclude the possibility of $\alpha_C = \pi/2$ (implying that $\beta_C = 0$) we define $\tan^{-1}\tan\pi/2 =: \pi/2$. We also use the value $\tan^{-1}\tan 0 =: 0$ determining the used range of the function $x \mapsto \tan^{-1} x$.

By the convexity (see e.g. p. 229 in [57]) of $\tan^{-1}(m\tan\alpha_A)$ we get that

$$\tau \leq 2\pi - 2\left(2\tan^{-1}\left(m\tan\frac{\alpha_A+\alpha_B}{2}\right) + \tan^{-1}(m\tan\alpha_C)\right).$$

On the other hand for m' we have $\tau = 2\pi - 2\left(2\tan^{-1}\left(m'\tan\frac{\alpha_A+\alpha_B}{2}\right) + \tan^{-1}(m'\tan\alpha_C)\right)$ implying that

$$\left(2\tan^{-1}\left(m\tan\frac{\alpha_A+\alpha_B}{2}\right) + \tan^{-1}(m\tan\alpha_C)\right) \leq \left(2\tan^{-1}\left(m'\tan\frac{\alpha_A+\alpha_B}{2}\right) + \tan^{-1}(m'\tan\alpha_C)\right)$$

from which it follows that $m' \geq m$.

Second assume that the angle at C is obtuse. Then $\alpha_A + \alpha_B = \alpha_C < \pi/2$ and we have

$$\tau = 2\left(\tan^{-1}(m\tan(\alpha_A+\alpha_B)) - \tan^{-1}(m\tan\alpha_A) - \tan^{-1}(m\tan\alpha_B)\right).$$

On the other hand $a(ABC) = \frac{1-m^2}{2}(\sin 2\alpha_A + \sin 2\alpha_B - \sin 2\alpha_C)$ and the volume in question is $v(\alpha_A,\alpha_B) = \frac{m(1-m^2)}{6}(\sin 2\alpha_A + \sin 2\alpha_B - \sin 2(\alpha_A+\alpha_B))$.

We consider the maximum of $v(\alpha_A,\alpha_B)$ under the conditions $0 \leq \alpha_A, \alpha_B \leq \pi/2$,

$$0 = -\frac{\tau}{2} + \left(\tan^{-1}(m\tan(\alpha_A+\alpha_B)) - \tan^{-1}(m\tan\alpha_A) - \tan^{-1}(m\tan\alpha_B)\right),$$

and $0 = \alpha_A + \alpha_B - \text{const}$, with respect to the unknown values α_A, α_B and m. Using Lagrange's method we get two equations

$$\mu = \frac{m(1-m^2)}{6}(\cos 2\alpha_A - \cos 2(\alpha_A+\alpha_B)) + \frac{\lambda m(1-m^2)\left(\tan^2(\alpha_A+\alpha_B) - \tan^2\alpha_A\right)}{\left(1+m^2\tan^2(\alpha_A+\alpha_B)\right)\left(1+m^2\tan^2\alpha_A\right)}$$

$$\mu = \frac{m(1-m^2)}{6}(\cos 2\alpha_B - \cos 2(\alpha_A+\alpha_B)) + \frac{\lambda m(1-m^2)\left(\tan^2(\alpha_A+\alpha_B) - \tan^2\alpha_B\right)}{\left(1+m^2\tan^2(\alpha_A+\alpha_B)\right)\left(1+m^2\tan^2\alpha_B\right)}$$

which are equivalent to the equations

$$\frac{\mu}{m(1-m^2)} = \frac{1}{3} + \frac{\lambda\left(1+\tan^2(\alpha_A+\alpha_B)\right)\left(1+\tan^2\alpha_A\right)}{\left(1+m^2\tan^2(\alpha_A+\alpha_B)\right)\left(1+m^2\tan^2\alpha_A\right)}$$

$$\frac{\mu}{m(1-m^2)} = \frac{1}{3} + \frac{\lambda\left(1+\tan^2(\alpha_A+\alpha_B)\right)\left(1+\tan^2\alpha_B\right)}{\left(1+m^2\tan^2(\alpha_A+\alpha_B)\right)\left(1+m^2\tan^2\alpha_B\right)}$$

because of the equality

$$\frac{\tan^2(\alpha_A+\alpha_B) - \tan^2\alpha_A}{\left(1+\tan^2(\alpha_A+\alpha_B)\right)\left(1+\tan^2\alpha_A\right)} = \cos^2\alpha_A - \cos^2(\alpha_A+\alpha_B) = \frac{\cos 2\alpha_A - \cos 2(\alpha_A+\alpha_B)}{2}.$$

These conditions turn out to be equivalent to

$$\frac{(1+\tan^2\alpha_A)}{(1+m^2\tan^2\alpha_A)} = \frac{(1+\tan^2\alpha_B)}{(1+m^2\tan^2\alpha_B)}$$

which cannot be satisfied unless $\alpha_A = \alpha_B$. Hence if the triangle is not an isosceles one it is not a local extremum of our problem, on the other hand by compactness it has at least one local maximum proving our statement. □

We can compare the formulas of Proposition 1.2.2

$$V \leq \frac{m'(1-m'^2)}{6}(2\sin\alpha_C - \sin 2(\alpha_C)) = \frac{m'(1-m'^2)}{3}\sin\alpha_C(1-\cos\alpha_C)$$

and

$$V \leq \frac{m'(1-m'^2)}{6}(2\sin(\pi-\widetilde{\alpha}_C) + \sin 2(\widetilde{\alpha}_C)) = \frac{m'(1-m'^2)}{3}\sin\widetilde{\alpha}_C(1+\cos\widetilde{\alpha}_C)$$

on α_C and $\widetilde{\alpha_C}$. In both cases we assumed that α_C and $\widetilde{\alpha}_C$ are in the interval $[0, \pi/2]$, respectively. Using the equality $\alpha_C = \pi - \widetilde{\alpha}_C$ the above formulas simplify to the following common form

$$V \leq \frac{m'(1-m'^2)}{3} \sin\alpha_C(1-\cos\alpha_C) =: v\,(m', \alpha_C) \quad \text{where} \quad 0 < \alpha < \pi. \tag{12}$$

In the case when $AC = BC$ we saw that $\tau = 2\left(\tan^{-1}\left(m' \tan\alpha_C\right) - 2\tan^{-1}\left(m' \tan\frac{\alpha_C}{2}\right)\right)$ and $\tau = 2\pi - 2\left(2\tan^{-1}\left(m' \tan\frac{\pi-\widetilde{\alpha_C}}{2}\right) + \tan^{-1}\left(m' \tan\widetilde{\alpha_C}\right)\right)$, respectively. (Observe that by the definition $\tan^{-1}(\infty) =: \pi/2$ these formulas are valid for $\alpha_C = \pi/2$ and lead to the same equality.) These equalities can be considered in the following common form

$$\tan\frac{\tau}{2} = \tan\left(\tan^{-1}\left(m' \tan\alpha_C\right) - 2\tan^{-1}\left(m' \tan\frac{\alpha_C}{2}\right)\right), \tag{13}$$

where $0 < \alpha_C < \pi$. In the case when $\pi/2 < \alpha_C$ we have $\tan(\tau/2) < 0$ and $\tau/2 = \pi + \tan^{-1}(\tan(\tau/2))$.

COROLLARY 1.2.1. *The upper bound function for fixed τ with the parameters $|AB|$,α_C is*

$$v(|AB|, \alpha_C) := \frac{|AB|^2}{12} \frac{\sqrt{\sin^2\alpha_C - \frac{|AB|^2}{4}}}{1+\cos\alpha_C}, \tag{14}$$

and using the equality $|AB| = 2\sin\frac{AB}{2}$ it is of the form

$$v(AB, \alpha_C) := \frac{\sin^2\frac{AB}{2}}{3} \frac{\sqrt{\sin^2\alpha_C - \sin^2\frac{AB}{2}}}{1+\cos\alpha_C}. \tag{15}$$

If AB is given the maximal volume of the possible facial tetrahedra are attained at the isosceles triangle with parameter value $\alpha_C = \cos^{-1}\left(\frac{|AB|^2}{4} - 1\right) = \cos^{-1}\left(-\cos^2\frac{AB}{2}\right)$. The formula is

$$v\left(|AB|, \cos^{-1}\left(\frac{|AB|^2}{4} - 1\right)\right) = \frac{|AB|}{6}\sqrt{\left(1 - \frac{|AB|^2}{4}\right)} = \frac{1}{6}\sin AB.$$

PROOF. Assume that the value of the length of AB is given. Then by Proposition 1.2.2 for fixed τ the maximal value of the volume V can be attained only for an isosceles triangle and the upper bound function gives this maximal volume. Using the equality

$$\sin\alpha_C = \frac{|AB|}{2\sqrt{1-m'^2}}$$

we get that

$$v\,(m', \alpha_C) = \frac{m'(1-m'^2)}{3}\sin\alpha_C(1-\cos\alpha_C) = \frac{|AB|^2}{12}\frac{\sqrt{\sin^2\alpha_C - \frac{|AB|^2}{4}}}{1+\cos\alpha_C} = v(|AB|, \alpha_C),$$

where the possible values of α_C can be get from the equality $\sin^2\alpha_C \geq |AB|^2/4$. The derivative of $v(|AB|, \alpha_C) = v(y, x)$ is

$$v'(y,x) = \frac{y^2\sin(x)\sqrt{\sin^2(x) - \frac{y^2}{4}}}{12(\cos(x)+1)^2} + \frac{y^2\sin(x)\cos(x)}{12(\cos(x)+1)\sqrt{\sin^2(x) - \frac{y^2}{4}}}$$

hence we have

$$v'(|AB|, \alpha_C) = \frac{|AB|^2\sin\alpha_C\left(\cos\alpha_C + 1 - \frac{|AB|^2}{4}\right)}{12(1+\cos\alpha_C)^2\sqrt{\sin^2\alpha_C - \frac{|AB|^2}{4}}} \quad \begin{cases} < 0 & \text{if} \quad \cos\alpha_C + 1 < \frac{|AB|^2}{4} \\ = 0 & \text{if} \quad \cos\alpha_C + 1 = \frac{|AB|^2}{4} \\ > 0 & \text{if} \quad \cos\alpha_C + 1 > \frac{|AB|^2}{4}. \end{cases}$$

Since $\cos^{-1}\left(\frac{|AB|^2}{4}-1\right)\leq\pi-\sin^{-1}(|AB|/2)$, on the interval

$$\sin^{-1}(|AB|/2)<\alpha_C\leq\pi/2\leq\cos^{-1}\left(\frac{|AB|^2}{4}-1\right)\leq\pi-\sin^{-1}(|AB|/2)$$

the function $v(\alpha_C)$ attains its maximal value at $\cos^{-1}\left(|AB|^2/4-1\right)$ furthermore

$$v\left(|AB|,\cos^{-1}\left(\frac{|AB|^2}{4}-1\right)\right)=\frac{|AB|^2}{12}\frac{\sqrt{\frac{|AB|^2}{4}\left(1-\frac{|AB|^2}{4}\right)}}{\frac{|AB|^2}{4}}=\frac{|AB|}{6}\sqrt{\left(1-\frac{|AB|^2}{4}\right)}.$$

$v(|AB|,\alpha_C)$ on the interval $\sin^{-1}(|AB|/2)<\alpha_C\leq\cos^{-1}\left(\frac{|AB|^2}{4}-1\right)$ is a strictly increasing function and on the interval $\cos^{-1}\left(\frac{|AB|^2}{4}-1\right)\leq\pi-\sin^{-1}(|AB|/2)$ it is a decreasing one. This shows that an optimal triangle with the fixed edge length $|AB|$ (which corresponding to a facial tetrahedron with maximal volume) is an isosceles one. □

We also have a formula on the upper bound function $v\left(m',\alpha_C\right)$ using as a parameter the surface area τ (introduced in Proposition 1.2.2).

PROPOSITION 1.2.3. [13] *Let the spherical area of the spherical triangle ABC be τ. Let α_C be the greatest central angle of ABC corresponding to AB. Then the volume V of the Euclidean pyramid with base ABC and apex O holds the inequality*

$$V\leq\frac{1}{3}\tan\frac{\tau}{2}\left(2-\frac{|AB|^2}{4}\left(1+\frac{1}{(1+\cos\alpha_C)}\right)\right). \tag{16}$$

In terms of τ and $c:=AB$ we have

$$V\leq v(\tau,c):=\frac{1}{6}\sin c\frac{\cos\frac{\tau-c}{2}-\cos\frac{\tau}{2}\cos\frac{c}{2}}{1-\cos\frac{c}{2}\cos\frac{\tau}{2}}. \tag{17}$$

Equality holds if and only if $|AC|=|CB|$.

PROOF. For $\alpha_C=\pi/2$ the statement is obviously true. In the other cases, by Proposition 2 and by the note before this statement we have to investigate the inequality

$$V\leq\frac{m'(1-m'^2)}{3}\sin\alpha_C(1-\cos\alpha_C)=:v\left(m',\alpha_C\right)\quad\text{where}\quad 0<\alpha_C<\pi,\quad\alpha_C\neq\pi/2$$

with the condition

$$\tan\frac{\tau}{2}=\tan\left(\tan^{-1}\left(m'\tan\alpha_C\right)-2\tan^{-1}\left(m'\tan\frac{\alpha_C}{2}\right)\right)=$$

$$\frac{m'\tan\alpha_C-\tan\left(2\tan^{-1}\left(m'\tan\frac{\alpha_C}{2}\right)\right)}{1+m'\tan\alpha_C\tan\left(2\tan^{-1}\left(m'\tan\frac{\alpha_C}{2}\right)\right)}=\frac{\frac{2m'\tan\frac{\alpha_C}{2}}{1-\tan^2\frac{\alpha_C}{2}}-\frac{2m'\tan\frac{\alpha_C}{2}}{1-m'^2\tan^2\frac{\alpha_C}{2}}}{1+\frac{2m'\tan\frac{\alpha_C}{2}}{1-\tan^2\frac{\alpha_C}{2}}\frac{2m'\tan\frac{\alpha_C}{2}}{1-m'^2\tan^2\frac{\alpha_C}{2}}}=$$

$$\frac{2m'(1-m'^2)\tan^3\frac{\alpha_C}{2}}{(1-\tan^2\frac{\alpha_C}{2})(1-m'^2\tan^2\frac{\alpha_C}{2})+4m'^2\tan^2\frac{\alpha_C}{2}}=\frac{2m'(1-m'^2)\tan\frac{\alpha_C}{2}}{(\cot\frac{\alpha_C}{2}-\tan\frac{\alpha_C}{2})(\cot\frac{\alpha_C}{2}-m'^2\tan\frac{\alpha_C}{2})+4m'^2}=$$

$$\frac{m'(1-m'^2)\sin\alpha_C(1-\cos\alpha_C)}{(1-m'^2)\left(\cos\alpha_C-\sin^2\alpha_C\right)+(1+m'^2)}=\frac{3v\left(m',\alpha_C\right)}{(1-m'^2)\cos\alpha_C(1+\cos\alpha_C)+2m'^2}.$$

Since

$$\sin\alpha_C=\frac{|AB|}{2\sqrt{1-m'^2}},$$

hence

$$1-m'^2=\frac{|AB|^2}{4\sin^2\alpha_C}$$

implying that

$$3v\left(m',\alpha_C\right)=\tan\frac{\tau}{2}\left(\frac{|AB|^2\cos\alpha_C(1+\cos\alpha_C)}{4\sin^2\alpha_C}+2\left(1-\frac{|AB|^2}{4\sin^2\alpha_C}\right)\right)=$$

$$\tan\frac{\tau}{2}\left(2+\frac{|AB|^2}{4\sin^2\alpha_C}\left(\cos\alpha_C(1+\cos\alpha_C)-2\right)\right)=\tan\frac{\tau}{2}\left(2-\frac{|AB|^2\left(2+\cos\alpha_C\right)}{4\left(1+\cos\alpha_C\right)}\right).$$

So

$$V\leq\frac{1}{3}\tan\frac{\tau}{2}\left(2-\frac{|AB|^2}{4}\left(1+\frac{1}{(1+\cos\alpha_C)}\right)\right)$$

as we stated.

Since $\pi-\alpha_C$ is the angle of the chordal triangle (rectilineal triangle) ABC at C, thus we can give it as a function of the spherical lengths of the sides of the spherical triangle ABC. Thus we have (see eq. (486) in [37])

$$\cos\alpha_C=-\frac{1+\cos AB-2\cos AC}{4\sin^2\frac{AC}{2}}=-\frac{-1+\cos AB+4\sin^2\frac{AC}{2}}{4\sin^2\frac{AC}{2}}.$$

Using the notation $a:=BC=AC$, $c=AB$ we get the formula

$$V\leq\frac{1}{3}\tan\frac{\tau}{2}\left(2-\sin^2\frac{AB}{2}-2\sin^2\frac{AC}{2}\right)=\frac{1}{3}\tan\frac{\tau}{2}\left(2-\sin^2\frac{c}{2}-2\sin^2\frac{a}{2}\right).$$

Finally use the spherical Heron's formula proved first by Lhuilier (see p.88 in [37]):

$$\tan\frac{\tau}{4}=\sqrt{\tan\frac{a+b+c}{4}\tan\frac{-a+b+c}{4}\tan\frac{a-b+c}{4}\tan\frac{a+b-c}{4}}.$$

Since $a=b$ it can be reduced to the form

$$\tan\frac{\tau}{4}=\tan\frac{c}{4}\sqrt{\tan\frac{2a+c}{4}\tan\frac{2a-c}{4}}=\tan\frac{c}{4}\sqrt{\frac{\sin^2\frac{a}{2}-\sin^2\frac{c}{4}}{1-\sin^2\frac{a}{2}-\sin^2\frac{c}{4}}}.$$

From this we get that

$$\sin^2\frac{a}{2}=\frac{\tan^2\frac{\tau}{4}\cos^2\frac{c}{4}+\tan^2\frac{c}{4}\sin^2\frac{c}{4}}{\tan^2\frac{\tau}{4}+\tan^2\frac{c}{4}}$$

and thus the inequality

$$V\leq\frac{1}{3}\tan\frac{\tau}{2}\left(2-\sin^2\frac{c}{2}-2\frac{\tan^2\frac{\tau}{4}\cos^2\frac{c}{4}+\tan^2\frac{c}{4}\sin^2\frac{c}{4}}{\tan^2\frac{\tau}{4}+\tan^2\frac{c}{4}}\right)=\frac{1}{3}\tan\frac{\tau}{2}\cos\frac{c}{2}\left(\cos\frac{c}{2}+\right.$$

$$\left.+\frac{\tan^2\frac{c}{4}-\tan^2\frac{\tau}{4}}{\tan^2\frac{c}{4}+\tan^2\frac{\tau}{4}}\right)=\frac{\sin\frac{\tau}{2}\cos\frac{c}{2}\sin^2\frac{c}{2}}{3\left(1-\cos\frac{c}{2}\cos\frac{\tau}{2}\right)}=\frac{\sin c\sin\frac{\tau}{2}\sin\frac{c}{2}}{6\left(1-\cos\frac{c}{2}\cos\frac{\tau}{2}\right)}=\frac{1}{6}\sin c\frac{\cos\frac{\tau-c}{2}-\cos\frac{\tau}{2}\cos\frac{c}{2}}{1-\cos\frac{c}{2}\cos\frac{\tau}{2}}.$$

□

REMARK 1.2.1. *In the case when $a=b=c$ the connection between the parameters c and τ is*

$$\tan\frac{\tau}{4}=\tan\frac{c}{4}\sqrt{\tan\frac{3c}{4}\tan\frac{c}{4}}=\tan^2\frac{c}{4}\sqrt{\frac{3-\tan^2\frac{c}{4}}{1-3\tan^2\frac{c}{4}}}.$$

To determine the parameter c we introduce the notion $x=\tan^2(c/4)$ and $\theta=\tan^2(\tau/4)$. Now we get the equation of order three

$$0=x^3-3x^2-3\theta x+\theta=(x-1)^3-3x(\theta+1)+(\theta+1),$$

and if we set $y=x-1$ then the equality

$$0=y^3-3y(\theta+1)-2(\theta+1).$$

Using Cardano's formula finally we get that

$$y=\frac{2\cos\left(\frac{\tau}{12}+\frac{4\pi}{3}\right)}{\cos\frac{\tau}{4}}.$$

Hence we have

$$\frac{1-\cos\frac{c}{2}}{1+\cos\frac{c}{2}}=\tan^2\frac{c}{4}=x=\frac{2\cos\left(\frac{\tau}{12}+\frac{4\pi}{3}\right)+\cos\frac{\tau}{4}}{\cos\frac{\tau}{4}}$$

implying that

$$\cos\frac{c}{2}=\frac{-1}{2\cos\frac{\tau+4\pi}{6}}\qquad\text{and}\qquad\sin^2\frac{c}{2}=\frac{4\cos^2\left(\frac{\tau+4\pi}{6}\right)-1}{4\cos^2\left(\frac{\tau+4\pi}{6}\right)}.$$

Substituting these values into the formula (17) we get the inequality of Proposition 1.2.1 showing that our result in the case of $p=3$ generalizes Prop. 1.2.1.

Assume now that the simplicial polyhedron P, starshaped with respect to the origin has f faces and is inscribed in the unit sphere. Let $c_1,\dots,c_f$ be the arc-lengths of the edges of the faces $F_1,\dots,F_f$ corresponding to their maximal central angles, respectively. Denote by τ_i the spherical area of the spherical triangle corresponding to the face F_i for all i. We note that for a spherical triangle which has edges a,b,c, the inequalities $0<a\le b\le c<\pi/2$ as well as the inequality $\tau\le c$ holds. In fact, for fixed τ the least value of the maximal edge length is attained at a regular triangle. If $c<\pi/2$ then we have

$$\tan\frac{\tau}{4}=\left(\tan\frac{c}{4}\sqrt{\tan\frac{3c}{4}\tan\frac{c}{4}}\right)=\left(\tan\frac{c}{4}\sqrt{1-\frac{\tan\frac{3c}{4}+\tan\frac{c}{4}}{\tan c}}\right)<\tan\frac{c}{4},$$

and if $c=\pi/2$ then $\tau=8\pi/4=\pi/2$ proving our statement.
Observe that the function $v(\tau,c)$ is concave in the parameter domain $\mathcal{D}:=\{0<\tau<\pi/2,\tau\le c<\min\{f(\tau),2\sin^{-1}\sqrt{2/3}\}\}$ with certain concave (in τ) function $f(\tau)$ defined by the zeros of the Hessian; and non-concave in the domain $\mathcal{D}'=\{0<\tau\le\omega,f(\tau)\le c\le 2\sin^{-1}\sqrt{2/3}\}=\{0<\tau\le c\le\pi/2\}\setminus D$, where $f(\omega)=2\sin^{-1}\sqrt{2/3}$. (The corresponding calculations can be checked by any symbolic software. The precise value of ω is approximately $\omega\approx 0.697715$.)

THEOREM 1.2.1. [13] *Assume that $0<\tau_i<\pi/2$ holds for all i. For $i=1,\dots,f'$ we require the inequalities $0<\tau_i\le c_i\le\min\{f(\tau_i),2\sin^{-1}\sqrt{2/3}\}$ and for all j with $j\ge f'$ the inequalities $0<f(\tau_j)\le c_j\le 2\sin^{-1}\sqrt{2/3}$, respectively. Let denote $c':=\frac{1}{f'}\sum_{i=1}^{f'}c_i$, $c^\star:=\frac{1}{f-f'}\sum_{i=f'+1}^{f}f(\tau_i)$ and $\tau':=\sum_{i=f'+1}^{f}\tau_i$, respectively. Then we have*

$$v(P)\le\frac{f}{6}\sin\left(\frac{f'c'+(f-f')c^\star}{f}\right)\frac{\cos\left(\frac{4\pi-f'c'-(f-f')c^\star}{2f}\right)-\cos\frac{2\pi}{f}\cos\left(\frac{f'c'+(f-f')c^\star}{2f}\right)}{1-\cos\frac{4\pi}{2f}\cos\left(\frac{f'c'+(f-f')c^\star}{2f}\right)}.\tag{18}$$

PROOF. The volume of P is bounded above by the quantity

$$v(P)\le\sum_{i=1}^{f}v(\tau_i,c_i):=\frac{1}{6}\sum_{i=1}^{f}\sin c_i\frac{\cos\frac{\tau_i-c_i}{2}-\cos\frac{\tau_i}{2}\cos\frac{c_i}{2}}{1-\cos\frac{c_i}{2}\cos\frac{\tau_i}{2}}.$$

Using the concavity of the function $v(\tau,c)$ on the domain $\mathcal{D}$ and the fact that the function $v(\tau,\cdot)$ for fixed τ is a monotone decreasing function of c on the domain $\mathcal{D}'$, we get the following upper bound for $v(P)$:

$$v(P)\le\frac{f'}{6}v\left(\frac{4\pi-\tau'}{f'},c'\right)+\frac{f-f'}{6}v\left(\frac{\tau'}{f-f'},c^\star\right).$$

Since for $i=f'+1,\dots,f$ the points $(\tau_i,f(\tau_i))$ are in the convex domain D then the point $\left(\frac{\tau'}{f-f'},c^\star\right)$ also in $\mathcal{D}$. Applying again the concavity property of the function $v(\tau,c)$, we get the inequality

$$v(P)\le\frac{f}{6}v\left(\frac{4\pi}{f},\frac{f'c'+(f-f')c^\star}{f}\right)=$$

$$\frac{f}{6}\sin\left(\frac{f'c'+(f-f')c^\star}{f}\right)\frac{\cos\left(\frac{4\pi-f'c'-(f-f')c^\star}{2f}\right)-\cos\frac{2\pi}{f}\cos\left(\frac{f'c'+(f-f')c^\star}{2f}\right)}{1-\cos\frac{4\pi}{2f}\cos\left(\frac{f'c'+(f-f')c^\star}{2f}\right)},$$

as we stated. □

REMARK 1.2.2. *When $f'=f$ we have the following formula:*

$$v(P)\le\frac{f}{6}\sin c'\frac{\cos\left(\frac{2\pi}{f}-\frac{c'}{2}\right)-\cos\frac{2\pi}{f}\cos\frac{c'}{2}}{1-\cos\frac{c'}{2}\cos\frac{2\pi}{f}},\tag{19}$$

where $c'=\frac{1}{f}\sum_{i=1}^{f}c_i$. In this case the upper bound is sharp if all face-triangles are obtuse isosceles ones with the same area and maximal edge lengths.

The condition of sharpness implies that the unit sphere tiling by the congruent copies of such isosceles spherical triangles which equal sides are less than or equal to the third one. Observe that a polyhedron corresponding to such a tiling could not be convex. This motivates the following problem: *Give such values τ and c that the isosceles spherical triangle with area τ and unique maximal edge length c can be generated by a tiling of the unit sphere.* We note that simplicial regular polyhedra satisfy this property.

EXAMPLE 1.2.1. We get a non-trivial example for this question, if we consider a rhombic dodecahedron with its centroid as the center of the sphere and we project from the center its vertices to the sphere (see the left figure in Fig.1.3). (Note that there is no circumscribed sphere about a rhombic dodecahedron hence the projection is necessary.) We get a tiling of the sphere containing congruent spherical quadrangles. One of these quadrangles has four congruent sides and two diagonals, respectively. The length of the longer diagonal is $c=\pi/2$.

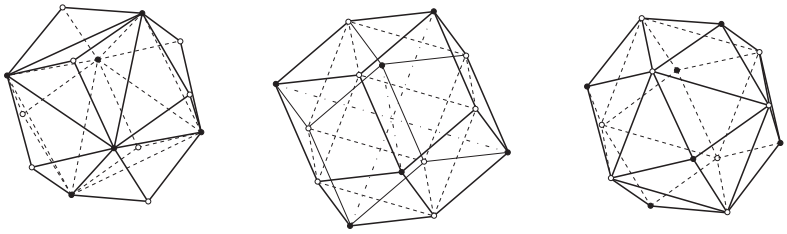

FIGURE 1.3. The star-shaped polyhedron P (on left), the original rhombic dodecahedron and the convex convex hull Q of P (on right).

We can dissect these quadrangles at these longer diagonals into two congruent spherical triangles. Denote by $\mathcal{P}$ the polyhedron defined by those plane triangles as facets which correspond to these spherical triangles, respectively. The angles and sides have the respective measures $\gamma=2\pi/3,\alpha=\pi/4,\beta=\pi/4$ and $c=\pi/2,a=\sin^{-1}\sqrt{2/3},b=\sin^{-1}\sqrt{2/3}$. Hence the area of this triangle $2\pi/3+\pi/2+\pi/2-\pi=\pi/6=4\pi/24$ as follows from the fact, that the 24 congruent copies of it, tile the whole sphere. Observe that $\mathcal{P}$ is not convex since the distance of the opposite vertices of two triangles with common base (in Euclidean measure) $(2/\sqrt{3})$ is less than that of the Euclidean length of the common base $(\sqrt{2})$. Since we have only one type of triangles for which $f(\tau_1)=f(\pi/6)\approx f(0,52360)\ge\pi/2=c_1$ we can apply (19) with $f=24$, $c'=\pi/2$, hence

$$v(\mathcal{P})=4\frac{\sqrt{2}\cos\frac{\pi}{6}-\cos\frac{\pi}{12}}{\sqrt{2}-\cos\frac{\pi}{12}}.$$

This quantity is an upper bound for the volume of such star-shaped polyhedra which are inscribed into the unit sphere, have 24 faces with spherical area τ_i with the assumption that $f(\tau_i)\ge\pi/2$ and with maximal edge length $\pi/2$. We get such polyhedra if we change a little

bit the position of those vertices of P which denoted by white circles on Fig. 1.3 (For τ (by Mathematica 10) we got the assumption $\pi/2 \geq \tau \geq \tan^{-1}\left(2\sqrt{5}-3\sqrt{2}\right)/(10+7\sqrt{2}) \approx \tau = 0.427922$.)
Denote by Q the convex hull of P (see the right figure on Fig. fig:starshaped.). Then $c_1 = 2\sin^{-1}\sqrt{1/3} \approx 1,23096 < \pi/2 < f(\tau_1)$ and we can apply again (19). Hence we get that

$$v(\mathcal{Q}) = \frac{8}{3}\frac{\sqrt{6}\cos\left(\frac{\pi}{12}-\sin^{-1}\sqrt{\frac{1}{3}}\right)-2\cos\frac{\pi}{12}}{\sqrt{3}-\cos\frac{\pi}{12}\sqrt{2}}.$$

Q has maximal volume of the class of such polyhedra which can be get from Q by a little change of the position of the vertices denoted by black circles, respectively.

EXAMPLE 1.2.2. Assume that $f' = f = 12$ and $c = 2\sin^{-1}(\sqrt{2/3})$. Then the upper bound is

$$2\frac{2\sqrt{2}}{3}\frac{\cos\left(\frac{\pi}{6}-\sin^{-1}(\sqrt{2/3})\right)-\frac{1}{\sqrt{3}}\cos\frac{\pi}{6}}{1-\frac{1}{\sqrt{3}}\cos\frac{\pi}{6}} = \frac{8}{3\sqrt{3}},$$

which is the volume of the cube inscribed into the unit sphere. Hence we got a new proof for that case of Theorem 3.3 of [81] when we restrict our examination to those triangulations in which there is no face-triangle having edge length greater than the edge length of a regular tetrahedron inscribed in the unit sphere.

We now apply our inequality to prove the general form of Theorem 3.3 in [81] in which the additional assumption "the tetrahedra are in dual position" has been omitted.

THEOREM 1.2.2. [13, 81] *Consider two regular tetrahedra inscribed in the unit sphere. The maximal volume of the convex hull P of the eight vertices is the volume of the cube C inscribed in to unit sphere, so*

$$v(P) \leq v(C) = \frac{8}{3\sqrt{3}}.$$

PROOF. We have to consider only that case which is not considered in [81]. Hence we assume that in the spherical regular triangles of the spherical tiling is corresponding to the first regular tetrahedron there are $2,1,1,0$ vertices of the second tetrahedron, respectively. The five points (the three vertices of the first spherical triangle and the two vertices of the second tetrahedron having in this triangle) having in the first closed spherical triangle form a triangular dissection of it into five other spherical triangle. Unfortunately, this dissection contains also such triangles which maximal edge lengths greater than that of the edge length of the regular spherical triangle containing them. On the other hand these triangles belong to the parameter domain $\mathcal{D}'$ (defined in Theorem 1.2.1) because $f(\pi/5) = 1.83487 < 2\sin^{-1}\sqrt{\frac{2}{3}}$. Hence the upper bound function for fixed τ is locally a decreasing function of c. So we can assume that all of these triangles have the same maximal spherical lengths, which is equal to $2\sin^{-1}\sqrt{\frac{2}{3}}$. Thus we get the following upper bound for the volume:

$$v(P) \leq v\left(\pi, 2\sin^{-1}\sqrt{\frac{2}{3}}\right) + 6v\left(\pi/3, 2\sin^{-1}\sqrt{\frac{2}{3}}\right) + \sum_{i=1}^{5} v\left(\tau_i, 2\sin^{-1}\sqrt{\frac{2}{3}}\right) =$$

$$\frac{1}{9} + \frac{4}{3\sqrt{3}} + \frac{2}{9}\sum_{i=1}^{5}\frac{\sin\frac{\tau_i}{2}}{\sqrt{3}-\cos\frac{\tau_i}{2}}$$

where $0 \leq \tau_i$ and $\sum_{i=1}^{5}\tau_i = \pi$. But with these conditions we have

$$\sum_{i=1}^{5}\frac{\sin\frac{\tau_i}{2}}{\sqrt{3}-\cos\frac{\tau_i}{2}} \leq 1.97836 < 2$$

implying that

$$v(P) < \frac{\frac{1}{\sqrt{3}} + 4 + \frac{4}{\sqrt{3}}}{3\sqrt{3}} < \frac{8}{3\sqrt{3}} = v(C)$$

as we stated. □

1.3. On the hyperbolic concept of volume

Our observation on the volume of hyperbolic orthoscemes concerns a deficiency in the two hundred years literature. Using hyperbolic orthogonal coordinates we discovered a formula on the volume of the orthosceme by its edge lengths. Of course it also manifests in a non-elementary integral however completes to a complete triplet of the collection of integrals derived by Lobachevsky and Bolyai, respectively. (The integral of Lobachevsky based on the dihedral angles of the orthosceme and the formulas of Bolyai used dihedral angles and edge lengths in a mixed form.) In this section we refer to the results of the paper [14].

In hyperbolic geometry to get the volume of a polyhedron has only one possibility. We have to transform the problem to a problem to calculate an appropriate integral. For this purpose we need methods to allowed the points with coordinates. We now give volume-integrals with respect to some important system of coordinates. We use that distance parameter k which introduced by J.Bolyai to express the curvature $K = \frac{-1}{k^2}$ of the hyperbolic space.

Consider in $\mathbb{H}^n$ a parasphere of dimension $n-1$ and its bundle of rays of parallel lines. Let ξ_n be the last coordinate axis, one of these rays, the origin will be the intersection of this line with the parasphere. The further $(n-1)$-"axes" are pairwise orthogonal paracycles. The coordinates of P in this system are $(\xi_1, \xi_2, \cdots, \xi_n)^T$, where the last coordinate is the distance of P and the parasphere, while the further coordinates are the coordinates of the orthogonal projection T with respect to the Cartesian coordinate system in $\mathbb{E}^{n-1}$ given by the above mentioned parasphere.

We can correspond to P a point p in $\mathbb{R}^n$ by Cartesian coordinates:

$$(x_1, x_2, \cdots, x_n)^T = \left(e^{-\frac{\xi_n}{k}}\xi_1, e^{-\frac{\xi_n}{k}}\xi_2, \cdots, e^{-\frac{\xi_n}{k}}\xi_{n-1}, \xi_n\right)^T.$$

By definition let the volume of a Jordan measurable set D in $\mathbb{H}^n$ be

$$v(D) := v_n \int_{D^\star} \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where $D^\star$ in $\mathbb{R}^n$ is the image of the domain D lying in $\mathbb{H}^n$ (by the above mapping) and v_n is a constant which we will choose later. Our first volume formula is:

$$v(D) = v_n \int_D e^{-(n-1)\frac{\xi_n}{k}} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n,$$

depending on the paracycle coordinates of the points of D, in the given system. Let now the domain $D = [0, a_1] \times \cdots \times [0, a_{n-1}]$ be a parasphere sector of parallel segments of length a_n based on a coordinate brick of the corresponding parasphere. Then we get by successive integration

$$v(D) = v_n \int_0^{a_1} \cdots \left[\int_0^{a_n} e^{-(n-1)\frac{\xi_n}{k}} \mathrm{d}\xi_n\right] \cdots \mathrm{d}\xi_1 = \frac{kv_n}{n-1}\left[-e^{-(n-1)\frac{a_n}{k}} + e^0\right] \prod_{i=1}^{n-1} a_i =$$

$$= \frac{kv_n}{n-1}[1 - e^{-(n-1)\frac{a_n}{k}}] \prod_{i=1}^{n-1} a_i.$$

If a_n tends to infinity and $a_i = 1$ for every $i = 1, \cdots, (n-1)$, then the volume is equal to $\frac{kv_n}{n-1}$. Note that J.Bolyai and N.I.Lobachevski used the value $v_n = 1$ only for $n = 2, 3$ so in their calculations the volume is independent of the dimension but depends on the constant k which determine the curvature of the space. To follow them we will determine the constant v_n such

that for every fixed k the measure of a thin layer divided by its height tends to the measure of the limit figure of lower dimension. Now the limit:

$$\lim_{a_n\to 0}\frac{v(D)}{a_n}=\frac{kv_n}{n-1}\lim_{n\to\infty}\frac{[1-e^{-(n-1)\frac{a_n}{k}}]}{a_n}\prod_{i=1}^{n-1}a_i=v_n\prod_{i=1}^{n-1}a_i,$$

would be equal to $v_{n-1}\prod_{i=1}^{n-1}a_i$ showing that $1=v_1=v_2=\ldots=v_n=\ldots$.
Thus $v_n=1$ as indicated earlier. On the other hand if for a fixed n the number k tends to infinity the volume of a body tends to the Euclidean volume of the corresponding Euclidean body. In every dimension n we also have a k for which the corresponding hyperbolic n-space contains a natural body with unit volume, if k equal to $n-1$ then the volume of the paraspheric sector based on a unit cube of volume 1 is also 1.
Finally, with respect to paracycle coordinate system our volume function by definition will be

$$v(D)=\int_D e^{-(n-1)\frac{\xi_n}{k}}\mathrm{d}\xi_1\cdots\mathrm{d}\xi_n.$$

Give now an orthogonal system $\mathcal{H}$ of axes associated to the paracycle coordinate system as follows. Let the new half-axes $x_1,\cdots,x_{n-1}$ be the tangent half-lines of the former paracycles at their common origin. (We can see the situation in Fig. 1.4) To determine the new coordinates of the point P we project P orthogonally to the hyperplane spanned by the axes $x_1,x_2,\cdots,x_{n-2},x_n$. The projection will be P_{n-1}. Then we project orthogonally P_{n-1} to the $(n-2)$-space is spanned by the axes $x_1,x_2,\cdots,x_{n-3},x_n$. The new point is P_{n-2}. Now the $(n-1)^{th}$ coordinate is the distance of P and P_{n-1}, the $(n-2)^{th}$ coordinate is the distance of P_{n-1} and P_{n-2} and so on .... In the last step we get the n^{th} coordinate which is the distance of the point P_1 from the origin O. Since the connection between the distance $2d$ of two points of a paracycle and the length of the connecting paracycle arc $2s$ is $s=k\sinh\frac{d}{k}$. Thus the dis-

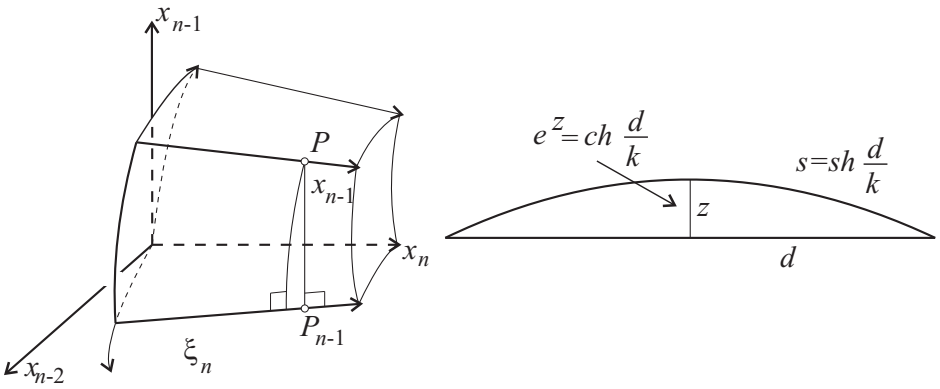

FIGURE 1.4. Coordinate system based on orthogonal axes

tance z of the respective halving points can be calculated as: $z=k\ln\cosh\frac{d}{k}$. Now a non-trivial but elementary calculation shows (using also the hyperbolic Pythagorean theorem) that the connection between the coordinates with respect to the two systems of coordinates is:

$$\begin{aligned}
\xi_{n-1} &= e^{\frac{\xi_n}{k}}k\sinh\frac{x_{n-1}}{k}\\
\xi_{n-2} &= e^{\frac{\xi_n}{k}+\ln\cosh\frac{x_{n-1}}{k}}k\sinh\frac{x_{n-2}}{k}\\
&\vdots\\
\xi_1 &= e^{\frac{\xi_n}{k}+\ln\cosh\frac{x_{n-1}}{k}+\cdots+\ln\cosh\frac{x_2}{k}}k\sinh\frac{x_1}{k}\\
x_n &= \xi_n+k\ln\cosh\frac{x_{n-1}}{k}+\cdots+k\ln\cosh\frac{x_2}{k}+k\ln\cosh\frac{x_1}{k}.
\end{aligned}$$

From this we get a new connection. Correspond the point $(\xi_1, \xi_2, \cdots, \xi_n)^T \in \mathbb{H}^n$ to the point $(u_1, \cdots, u_n)^T \in \mathbb{R}^n$ as in our first calculation. The corresponding system of equations is:

$$\begin{aligned} u_1 &= k \cosh \frac{x_2}{k} \cdots \cosh \frac{x_{n-1}}{k} \sinh \frac{x_1}{k} \\ &\vdots \\ u_{n-1} &= k \sinh \frac{x_{n-1}}{k} \\ u_n &= x_n - k \ln \cosh \frac{x_1}{k} - \cdots - k \ln \cosh \frac{x_{n-1}}{k} \end{aligned}$$

The Jacobian determinant of this transformation is

$$\left(\cosh \frac{x_1}{k}\right) \left(\cosh \frac{x_2}{k}\right)^2 \cdots \left(\cosh \frac{x_{n-1}}{k}\right)^{n-1},$$

implying our second formula on the volume:

$$v(D) = \int_D \left(\cosh \frac{x_{n-1}}{k}\right)^{n-1} \cdots \left(\cosh \frac{x_2}{k}\right)^2 \left(\cosh \frac{x_1}{k}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

The *orthoscheme* is a special tetrahedron. Two edges a and b are orthogonal to each other and a third one c (skew to a) is orthogonal to the plane of a and b (and intersects b). Let x the third edge of the triangle with edges a and b, y the third edge of the triangle of b and c and z the remaining side of the orthosceme. The dihedral angle at a is α, the angle opposite to b of the triangle with edges a and b is β and the angle opposite to the edge c in the triangle with edges c and z is γ, respectively. J.Bolyai gave two formulas on the volume ($k = 1$) (see in [29], [143]):

$$v = \frac{\tan \gamma}{2 \tan \beta} \int_0^c \frac{u \sinh u}{\left(\frac{\cosh^2 u}{\cos^2 \alpha} - 1\right) \sqrt{\frac{\cosh^2 u}{\cos^2 \gamma} - 1}} \mathrm{d}u,$$

and

$$v = \frac{1}{2} \int_0^\alpha \left(-a + \frac{\sinh a \cos \phi}{2\sqrt{\tanh^2 b + \sinh^2 a \cos^2 \phi} \ln \frac{\cosh a \cos \phi + \sqrt{\tanh^2 b + \sinh^2 a \cos^2 \phi}}{\cosh a \cos \phi - \sqrt{\tanh^2 b + \sinh^2 a \cos^2 \phi}}} \right) \mathrm{d}\phi$$

For the so-called *asymptotic orthoscheme* for which the ideal vertex is the common endpoint of the edges a, x and z it gives the formulas:

$$v = \frac{\sin 2\alpha}{4} \int_0^c \frac{u}{\cosh^2 u - \cos^2 \alpha} \mathrm{d}u \text{ and } v = \frac{1}{2} \int_0^\alpha \ln \frac{\cos \phi}{\sqrt{\cos^2 \phi - \tanh^2 b}} \mathrm{d}\phi,$$

respectively.

The formula of Lobachevsky can be get as follows. Let the essential (non-rectangular) dihedral angles of an orthoscheme be α, β and γ. They are admitted to the edges a, z and c, respectively. Introduce the parameter δ by the equalities:

$$\tanh \delta := \tanh a \tan \alpha = \tanh c \tan \gamma,$$

and the Milnor's form of the Lobachevsky-function (see in [122])

$$\Lambda(x) = -\int_0^x \ln |2 \sin \xi| \mathrm{d}\xi,$$

respectively. Then the volume v of the orthoscheme in the case of $k = 1$ is

$$\frac{1}{4} \left[\Lambda(\alpha + \delta) - \Lambda(\alpha - \delta) - \Lambda\left(\frac{\pi}{2} - \beta + \delta\right) + \Lambda\left(\frac{\pi}{2} - \beta - \delta\right) + \right.$$

$$+\Lambda(\gamma+\delta)-\Lambda(\gamma-\delta)+2\Lambda\left(\frac{\pi}{2}-\delta\right)\Big].$$

As an application of our general formulas we determine the volume of the orthosceme as the function of its edge-lengthes a, b and c. We note that the formulas

$$a=\frac{1}{2}\ln\frac{\sin(\alpha+\delta)}{\sin(\alpha-\delta)},\quad c=\frac{1}{2}\ln\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)},\quad z=\frac{1}{2}\ln\frac{\sin(\frac{\pi}{2}-\beta+\delta)}{\sin(\frac{\pi}{2}-\beta-\delta)}.$$

transform the dihedral angles into the edge-lengthes. This observation gives another possibility to get our formula from the classical ones but the corresponding calculation seems to be very uncomfortable.

The following lemma in the three-dimensional case can be proved easily.

LEMMA 1.3.1. [14] *We have two k-dimensional hyperbolic subspaces H_k and H'_k, respectively for which they intersection has dimension $k-1$. Assume that the points $P\in H_k$, $P'\in H'_k$ and $P''\in H_k\cap H'_k$ hold the relations $PP'\perp H'_k$ and $P'P''\perp H_k\cap H'_k$, respectively. Then the angle*

$$\alpha=\tan^{-1}\frac{\tanh(PP')}{\sinh P'P''},$$

is independent from the position of P in H_k.

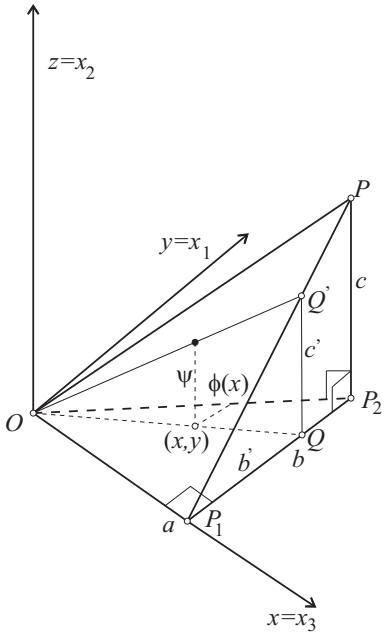

FIGURE 1.5. Orthoscheme and orthogonal coordinates

For our purpose we have to determine the integral

$$v(D)=\int_D(\cosh z)^2(\cosh y)\mathrm{d}z\mathrm{d}y\mathrm{d}x=\int_0^a\left[\int_0^{\phi(x)}\left(\int_0^{\psi(x,y)}(\cosh z)^2(\cosh y)\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}x.$$

which is based on hyperbolic orthogonal coordinates (by $k=1$). In this formula the functions $\phi(x)$ and $\psi(x,y)$ can be determined as follows. Consider the orthoscheme in Figure 1.5. In the rectangular triangle $\triangle_{OP_2P_1}$ we know that the tangent of the angle $P_2OP_1\angle$ is:

$$\tan P_2OP_1\angle=\frac{\tanh b}{\sinh a}=\frac{\tanh\Phi(x)}{\sinh x}.$$

Hence

$$\tanh\Phi(x)=\frac{\tanh b}{\sinh a}\sinh x,\ \text{and}\ 0\le y\le\phi(x)=\tanh^{-1}\left(\frac{\tanh b}{\sinh a}\sinh x\right)=:\lambda.$$

Consider now the triangle $\triangle_{P_1P_2P}$. The line $O(x,y,0)$ intersects that point Q for which $|P_1Q| = b'$, and let denote the point of the segment PP_1 above Q be Q'. Thus we get the equality

$$\tanh c' = \frac{\tanh c}{\sinh b}\sinh b'.$$

Take into consideration again the equality

$$\tanh b' = \frac{\tanh y}{\sinh x}\sinh a,$$

and apply the hyperbolic Pythagorean theorem. From the triangle $\triangle_{OQQ'}$ we get

$$\tanh\Psi(x,y) = \tanh c'\left[\frac{\sinh\left(\cosh^{-1}(\cosh x\cosh y)\right)}{\sinh\left(\cosh^{-1}(\cosh a\cosh b')\right)}\right] =$$

$$= \frac{\tanh c}{\sinh b}\sinh b'\left[\frac{\sqrt{\cosh^2 x\cosh^2 y - 1}}{\sqrt{\cosh^2 a\cosh^2 b' - 1}}\right] = \frac{\tanh c}{\sinh b}\sinh b'\left[\frac{\sqrt{\sinh^2 y + \sinh^2 x\cosh^2 y}}{\sqrt{\sinh^2 b' + \sinh^2 a\cosh^2 b'}}\right] =$$

$$= \frac{\tanh c}{\sinh b}\sinh y\frac{\sqrt{1+\sinh^2 x\coth^2 y}}{\sqrt{1+\sinh^2 a\coth^2 b'}} = \frac{\tanh c}{\sinh b}\sinh y,$$

since

$$\tan QOP_2\angle = \frac{\tanh b'}{\sinh a} = \frac{\tanh y}{\sinh x}.$$

Hence the assumption

$$0 \le z \le \psi(x,y) = \tanh^{-1}\left(\frac{\tanh c}{\sinh b}\sinh y\right) =: \mu$$

holds if we fix the first two variables, but $\Psi(x,y)$ does not depend on x, as it can be expected in Lemma 1.3.1. Thus the desired volume is:

$$v = \int_0^a\int_0^\lambda\int_0^\mu (\cosh z)^2(\cosh y)\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^a\int_0^\lambda \frac{1}{2}\left[z+\frac{1}{2}(\sinh 2z)\right]_0^\mu(\cosh y)\mathrm{d}y\mathrm{d}x.$$

For $\Phi(x)$ and $\Psi(x,y)$ we apply the identities $\tanh\rho = \frac{\sinh\rho}{\cosh\rho} = \frac{e^{2\rho}-1}{e^{2\rho}+1}$, i.e. $\rho = \frac{1}{2}\ln\frac{1+\tanh\rho}{1-\tanh\rho}$. We get $\mu = \frac{1}{2}\ln\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}$, and $\lambda = \frac{1}{2}\ln\frac{\sinh a+\tanh b\sinh x}{\sinh a-\tanh b\sinh x}$. Hence

$$v = \frac{1}{4}\left\{\int_0^a\left(\int_0^\lambda \ln\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}\cosh y\mathrm{d}y + \right.\right.$$

$$\left.\left. + \int_0^\lambda \sinh\left(\ln\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}\right)\cosh y\mathrm{d}y\right)\mathrm{d}x\right\}.$$

To determine the second integral, we apply $\sinh u = \frac{e^u-e^{-u}}{2}$. Now

$$\int_0^\lambda \sinh\left(\ln\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y}\right)\cosh y\mathrm{d}y =$$

$$= \frac{1}{2}\int_0^\lambda\left(\frac{\sinh b+\tanh c\sinh y}{\sinh b-\tanh c\sinh y} - \frac{\sinh b-\tanh c\sinh y}{\sinh b+\tanh c\sinh y}\right)\cosh y\mathrm{d}y =$$

$$= 2\int_0^\lambda \frac{\sinh y\cosh y}{\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\sinh^2 y}\mathrm{d}y = 2\int_0^\lambda \frac{\sinh 2y}{2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2y + \frac{\tanh c}{\sinh b}}\mathrm{d}y =$$

$$= -\frac{\sinh b}{\tanh c}\left[\ln\left(2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2y + \frac{\tanh c}{\sinh b}\right)\right]_0^\lambda =$$

$$= -\frac{\sinh b}{\tanh c}\ln\left(2\frac{\sinh b}{\tanh c} - \frac{\tanh c}{\sinh b}\cosh 2\lambda + \frac{\tanh c}{\sinh b}\right) + \frac{\sinh b}{\tanh c}\ln\left(2\frac{\sinh b}{\tanh c}\right).$$

From the above expression of $\lambda = \Phi(x)$ we can calculate $\cosh 2\lambda$ and get:

$$\cosh 2\lambda = \frac{1}{2}\left(\frac{\sinh a + \tanh b \sinh x}{\sinh a - \tanh b \sinh x} + \frac{\sinh a - \tanh b \sinh x}{\sinh a + \tanh b \sinh x}\right).$$

Thus the second integral (denoted by II) is:

$$\mathrm{II} := -\frac{\sinh b}{\tanh c}\ln\left(1 - \frac{\tanh^2 c \sinh^2 x}{\cosh^2 b(\sinh^2 a - \tanh^2 b \sinh^2 x)}\right).$$

The first integral to v can be integrated by parts as follows:

$$\int_0^\lambda \ln\left(\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}\right)\cosh y \mathrm{d}y = \left\{\left[\ln\left(\frac{\sinh b + \tanh c \sinh y}{\sinh b - \tanh c \sinh y}\right)\sinh y\right]_0^\lambda -\right.$$

$$\left. - \int_0^\lambda \frac{\tanh c \cosh y[(\sinh b - \tanh c \sinh y) + (\sinh b + \tanh c \sinh y)]}{\sinh^2 b - \tanh^2 c \sinh^2 y}\sinh y \mathrm{d}y\right\} =$$

$$= \left\{\sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda} - \int_0^\lambda \frac{2\tanh c \sinh b \cosh y \sinh y}{\sinh^2 b - \tanh^2 c \cosh^2 y + \tanh^2 c}\mathrm{d}y\right\} =$$

$$= \left\{\sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda} + \frac{\sinh b}{\tanh c}\left[\ln(\sinh^2 b - \tanh^2 c \cosh^2 y + \tanh^2 c)\right]_0^\lambda\right\} =$$

$$= \left\{\sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda} + \frac{\sinh b}{\tanh c}\left(\ln(\sinh^2 b - \tanh^2 c \sinh^2\lambda) - \ln(\sinh^2 b)\right)\right\}.$$

Since

$$\sinh^2\lambda = \frac{\tanh^2 b \sinh^2 x}{\sinh^2 a - \tanh^2 b \sinh^2 x},$$

the first integral is:

$$\left\{\sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda} + \frac{\sinh b}{\tanh c}\ln\left(1 - \frac{\tanh^2 c \sinh^2 x}{\cosh^2 b(\sinh^2 a - \tanh^2 b \sinh^2 x)}\right)\right\} =$$

$$= \sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda} - \mathrm{II}.$$

The sum of the two parts is:

$$\sinh\lambda \ln\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}.$$

A change of integration variable will have some benefits $x \mapsto \lambda$, $a \mapsto b$, $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}\lambda}\mathrm{d}\lambda$. From $\lambda = \tanh^{-1}\left(\frac{\tanh b}{\sinh a}\sinh x\right)$ follows

$$x = \sinh^{-1}\left(\frac{\tanh\lambda \sinh a}{\sinh b}\right) = \ln\frac{\tanh\lambda \sinh a + \sqrt{\tanh^2\lambda \sinh^2 a + \tanh^2 b}}{\tanh b}$$

and we get in a straightforward way

$$v = \frac{1}{4}\int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}}\ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right)\mathrm{d}\lambda,$$

proving our main theorem as follows:

THEOREM 1.3.1. [14] *Let the edges of an orthoscheme be a, b, c, respectively, where $a \perp b$ and $(a, b) \perp c$. If $k = 1$ then its volume is:*

$$v = \frac{1}{4} \int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda.$$

COROLLARY 1.3.1. *This formula can be simplified in the case of asymptotic orthoschemes. If the edge-length a tends to infinity, the function $\frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}}$ tends to $\frac{1}{\cosh\lambda}$ showing that the volume of the orthosceme with one ideal vertex is*

$$v = \frac{1}{4} \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda.$$

If the length of the edge c also grows to infinity, then this formula simplifies to:

$$v = \frac{1}{4} \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda,$$

which is the volume of an orthosceme with two ideal vertices. If now we reflect this one in the face containing the edges b and c then we get a tetrahedron with three ideal vertices. If then we reflect the previous tetrahedron in the face containing the edges b and a we get another one with four ideal vertices. The volume of the last one is

$$v = \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda.$$

This tetrahedron has two edges (a and c) which are skew and orthogonal to each other (its common normal transversal is b). Since the reflection in the line of b is a symmetry of this ideal tetrahedron, we can see that there are two types of its dihedral angles, two opposite (at the edges a and c) are equal to each other, (say A); and the other four ones are also equal to each other (say B). Then we have $A + 2B = \pi$, and its volume by Milnor's formula is equal to

$$v' = \Lambda(\pi - 2B) + 2\Lambda(B) = \Lambda(2B) + 2\Lambda(B) = 4\Lambda(B) + 2\Lambda\left(B + \frac{\pi}{2}\right).$$

(We have exploited that the Lobachevsky function is odd, of period π, and satisfies the identity $\Lambda(2B) = 2\Lambda(B) + 2\Lambda(B + \frac{\pi}{2})$.) Then we get the following connection between the two integrals:

$$0 = \int_0^b \frac{1}{\cosh\lambda} \ln\left(\frac{\sinh b + \sinh\lambda}{\sinh b - \sinh\lambda}\right) \mathrm{d}\lambda + 2 \int_0^{B+\frac{\pi}{2}} \ln|2\sin\xi| \mathrm{d}\xi + 4 \int_0^B \ln|2\sin\xi| \mathrm{d}\xi.$$

If we substitute into our formula the first-order terms of the Taylor series of the functions in the integrand, respectively, we get

$$v = \frac{1}{4} \int_0^b \frac{\tanh\lambda \sinh a}{\sqrt{\tanh^2 b \cosh^2\lambda + \sinh^2 a \sinh^2\lambda}} \ln\left(\frac{\sinh b + \tanh c \sinh\lambda}{\sinh b - \tanh c \sinh\lambda}\right) \mathrm{d}\lambda =$$

$$= \frac{1}{2} \int_0^b \frac{\lambda a}{\sqrt{b^2 + a^2\lambda^2}} \frac{c\lambda}{b} \mathrm{d}\lambda = \frac{ac}{2b^2} \int_0^b \frac{\lambda^2}{\sqrt{1}} \mathrm{d}\lambda = \frac{abc}{6}.$$

This shows that it gives back the Euclidean volume for infinitesimal values.

CHAPTER 2

Investigations in a classical Minkowski normed space

2.1. Bisectors

The present dissertation refers to *bisectors* in (finite dimensional normed or) *Minkowski spaces*, i.e., to collections of points which have, in each case, the same distance (with respect to the corresponding norm) to two given points x, y of these spaces. Note that bisectors in Minkowski spaces play an essential role in Discrete and Computational Geometry, mainly in view of constructing (generalized) Voronoi diagrams, and also for motion planning with respect to translations; see, e.g., the surveys [19] and [116].

2.1.1. Bisectors and the unit ball. If K is a 0-symmetric, bounded, convex body in the Euclidean n-space E^n (with a fixed origin O) then it defines a norm whose unit ball is K itself (see [77] or [132]). Such a space is called *Minkowski normed space*. In fact, the norm is a continuous function which is considered (in the geometric terminology as in [77]) gauge function. The metric (the so-called Minkowski metric), the distance of two points, induced by this norm, is invariant with respect to the translations of the space.
The unit ball is said to be *strictly convex* if its boundary contains no line segment. A body is said to be *smooth* if each point on its boundary has a unique supporting hyperplane. There are dual notions with respect to the scalar product of the embedding Euclidean space. The dual body K^* of K is

$$K^* = \{y | \langle x, y\rangle \leq 1 \text{ for all } x \in K\}$$

where $\langle \cdot, \cdot \rangle$ means the inner product of the embedding Euclidean space. It can be shown (see [41]) that the (convex) unit ball K is strictly convex if and only if its dual body K^* is smooth. We examined in [1] the boundary of the unit ball of the norm and give two theorems similar to the characterization of the Euclidean norm investigated by H.Mann, A.C.Woods and P.M.Gruber in [111], [147], [74], [75] and [76], respectively. H.Mann proved that a Minkowskian normed space is Euclidean one (so its unit ball is an ellipsoid) if and only if all *Leibnizian halfspaces* (containing those points of the space which are closer to the origin than to another point x) are convex. A.C.Woods proved the analogous statement for such a distance function whose unit ball is bounded but is not necessarily centrally symmetric or convex. P.M Gruber extended the theorem for distance functions whose unit ball is a ray set. P.M. Gruber generalized the Woods' theorem in another way, too. He showed (see Satz.5 in [74]) that a bounded distance function gives a Euclidean norm if and only if there is a subset T of the $(n-1)$-dimensional unit sphere whose relative interior (with respect to the sphere) is not empty, having the property: each of the pairs of points {0,x} where $x \in T$ the corresponding Leibnizian halfspace is convex. From the convexity of the Leibnizian halfspaces follows that the collection of all points of the space whose distances from two distinct points are equal are hyperplanes. We call such a set the *bisector* of the considered points. Thus from Mann's theorem follows a theorem stated first explicitly by M.M.Day in [42]:

THEOREM 2.1.1 ([42]). *All of the bisectors, with respect to the Minkowski norm defined by the body K, are hyperplanes if and only if K is an ellipsoid.*

Day pointed out that this result is an immediate Corollary of a result of James [92].
We note that Day's theorem is also a consequence of a third (ellipsoid characterization) theorem proved by P.M.Gruber ([75] Satz.3) which says that if K_1 is a convex body in E^d ($d \geq 3$), and the intersection of the boundaries of the bodies K_2' and K_1 is contained in a hyperplane for all

translates K_2' of K_1 with $K_2' \neq K_1$ then K_1 is an ellipsoid. P.R.Goodey gave a little bit more general form of this theorem in ([67] and [68]), showing that if K_2 is another convex body of the space as K_1, and the intersection of the boundaries of the bodies K_2' and K_1 is contained in a hyperplane for all translates K_2' of K_2 with $K_2' \neq K_1$, then K_1 and K_2 are homothetic ellipsoids.
The second question concerning Day's theorem also was posed by H.Mann in [111]. He proved that if for all lattices of the embedding space the closed Dirichlet-Voronoi cell of a lattice point (determined by the Minkowski norm) is convex (in the usual Euclidean sense) then the norm is Euclidean one, too. This theorem was also extended by P.M.Gruber for a distance function with bounded star-shaped unit ball.
It is possible that the interior (with respect to the Minkowski metric) of a Dirichlet-Voronoi cell is convex while the closed one is not, thus we have to distinguish the open and the closed Dirichlet-Voronoi cells from each other. The "walls" such a closed cell may be an n-dimensional set in the Euclidean n-space. It is also possible that the bisector of $\{0, x\}$ is an n-dimensional part of the space. This is the case, e.g., if the unit ball is a square of the plane and the vector x is parallel to one of the edges of this square.

DEFINITION 2.1.1. *The* bisector *of the segment, corresponding to the position vector x, is*

$$H_x := \{y \in E^n \quad | \quad \|y\|_K = \|y - x\|_K\}.$$

We denote by $H_{x,0}$ and $H_{x,x}$ the Leibnizian halfspaces *to the segments $[0, x]$ and $[x, 0]$, respectively, as the set of those points which are closer (with respect to the norm $\|\cdot\|_K$) to the first end than to the second one.*

It is clear that if $\mathrm{cl}_K S$ denotes the closure of the set S with respect to the norm $\|\cdot\|_K$ we have

$$H_x = \mathrm{cl}_K H_{x,0} \cap \mathrm{cl}_K H_{x,x}.$$

Now, we prove some properties of the Leibnizian halfspaces and the bisectors.

LEMMA 2.1.1 ([1]). *With respect to the Euclidean metric topology of the embedding n-space the following properties hold:*

(1) H_x is a closed, connected set which is convex in the direction of the vector x, i.e. if a line parallel to x intersects H_x in two distinct points, then the whole segment with these endpoints also belongs to H_x.
(2) $H_{x,0}$ and $H_{x,x}$ are open, connected sets separated by the bisector H_x.

PROOF. From the continuity of the norm function it is easy to prove that the sets

$$H_{x,0} := \{y \in E^n | \|y\|_K < \|x - y\|_K\}$$
$$H_{x,x} := \{y \in E^n | \|y\|_K > \|x - y\|_K\}$$

are open with respect to the Euclidean metric topology, too. This means that H_x is closed.
Using the triangle inequality (by the convexity of K) it is easy to see that $H_{x,0}$ is a star-shaped set. This means that it is connected, too.
Prove now that H_x is convex in the direction of x. Let y and z be two points of H_x for which $y - z$ parallel to x and $\|y\|_K \geq \|z\|_K$. Consider the points $u = y - z$, $v = y - z + x$, 0 and x. If $\|y\|_K < \|z\|_K$ (see Figure 2.1) then we have

$$\|u - y\|_K = \|v - y\|_K = \|z - x\|_K = \|z\|_K > \|y\|_K = \|0 - y\|_K = \|x - y\|_K.$$

Thus u, v are on the boundary of the Minkowski ball with center y and radius $\|z\|_K$, while the points 0 and x are in the interior of this ball. This means that the points $u, v, 0, x$ in their line must have the order $[u, 0, x, v]$. It is impossible because $v - u = x$. From this we get that $\|y\|_K = \|z\|_K$. Let now E, F, E', F' be the ends of the position vectors y, z, $y - x$ and $z - x$, respectively.
These points are on the boundary of the K-ball with center 0 and radius $\|y\|_K$ which means that the segment $\mathrm{conv}\{E, F, E', F'\}$ belongs to the boundary of this ball. (At least three of these

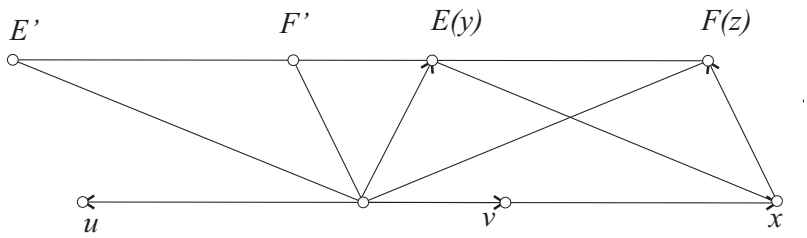

FIGURE 2.1. The proof of directional convexity

points are distinct.) So the intersection of the considered line with the bisector H_x contains the segment $\overline{EF}$ as we stated.
Since the intersection of a line l parallel to x with a closed K-ball is a compact segment, if we consider another K-ball K_1 intersecting the line l, the following non-empty set

$$(K_1 \cup (K_1 + x)) \cap l$$

is also compact. The complement of this set on the line l contains two open half lines l_- and l_+ satisfying the properties that the points of K_1 separate the points of l_- from the right endpoint of $(K_1 + x) \cap l$ and the points of $K_1 + x$ separate the points of l_+ from the left endpoint of $K_1 \cap l$, respectively. It is easy to see that the points of l_- belong to $H_{x,0}$ and the points of l_+ belong to $H_{x,x}$, respectively. So by the continuity of the Minkowski norm, every line parallel to x can be divided into three non-empty parts: a compact segment (may be degenerated to a point) belongs to H_x and two open halflines belong to $H_{x,0}$ and $H_{x,x}$, respectively.
Consider now a hyperplane orthogonal to the vector x and take the orthogonal projection of H_x into this $(n-1)$-dimensional Euclidean space. If we assume that H_x can be decomposed into the union of two disjoint closed subsets of it, then the images of these components (by the convexity in the direction of x and the above trisection of any projection line) are disjoint closed subsets whose union is this hyperplane. Using now the connectivity of the hyperplane we get that this decomposition is trivial and in fact H_x is connected, too.
The last statement of this lemma is the separating property of the bisector. Consider an elementary curve γ which connects a point y of $H_{x,0}$ with a point z of $H_{x,x}$. Since $H_{x,0}$ and $H_{x,x}$ are open with respect to the Euclidean topology of the space, the sets $H_{x,0} \cap \gamma$ and $H_{x,x} \cap \gamma$ are open in the induced topology of the connected curve γ. However, these sets are non-empty and disjoint hence there is (at least one) point of γ which lies in the complement of $H_{x,0} \cup H_{x,x}$, i.e. in H_x. So for every pairs of such points y, z and their connecting curve γ there is a point of $\gamma \cap H_x$ which separates the endpoints of γ. □

The results of the following two lemmas seem to be new. The first one is an important consequence of the statements of Lemma 2.1.1.

LEMMA 2.1.2 ([1]). *The boundary of K does not contain any line segment parallel to x if and only if for each line l parallel to x the set*

$$H_x \cap l$$

contains exactly one point.

PROOF. Assume indirectly that the boundary of K, denoted by $\operatorname{bd} K$, contains a non-degenerate segment s parallel to x (see Figure 2.2).For the line l containing s we have $\operatorname{bd} K \cap l = s$ and $(\operatorname{bd} K + x) \cap l = s + x$. This means that for a sufficiently large real number r the set $\operatorname{bd}(rK) \cap \operatorname{bd}(rK) + x$ contains the non-degenerate segment $rs \cap rs + x$. This proves one direction of the lemma.
Conversely, if $H_x \cap l$ contains the points y and z then as we saw in the proof of the convexity part of the proof of Lemma 2.1.1 (to Figure 2.1), the following equalities hold

$$\|y\|_K = \|z\|_K = \|y - x\|_K = \|z - x\|_K,$$

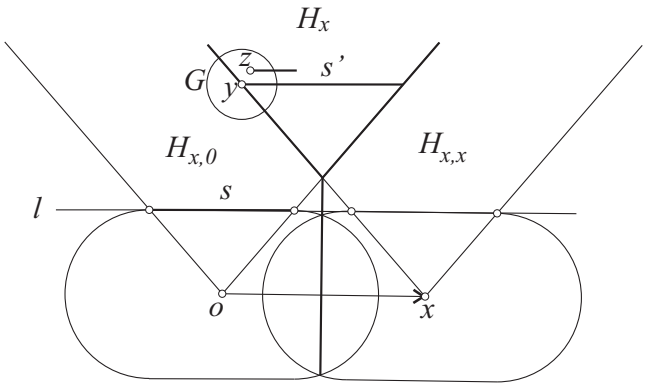

FIGURE 2.2. Maximal segment s' in H_x.

which means again that the set $H_x \cap l$ contains at least three distinct points of the boundary of the K-ball with center 0 and radius $\|y\|_K$. This means that the boundary of this ball contains a segment parallel to x which proves our Lemma 2.1.1. □

Our last lemma formulates a topological property of the bisector. We shall use the natural notion of *maximal segment s' belonging to H_x parallel to x* and the *left or right end* of s'. (Left end of s' is from which any other point of s' can be get by adding a positive multiples of x.) It is possible that a left end of a maximal segment belonging to H_x is an inner point of the closed set $\mathrm{cl}_{E^n} H_{x,0}$ meaning that there exists an open Euclidean n-ball G around this left end which does not intersect the other Leibnitzian halfspace $H_{x,x}$. We prove that in this case the bisector does not a topological hyperplane.

LEMMA 2.1.3 ([1]). *Let $y \in H_x$ be a left end of a maximal segment s' belonging to H_x parallel to x and having non-zero length. If there is an n-dimensional open Euclidean ball G with center y for which $H_{x,x} \cap G$ is empty then H_x does not homeomorphic to a hyperplane.*

Before the proof of this lemma we recall the definition of topological manifold with relative boundary points. An $(n-1)$-dimensional *topological manifold* is a separable topological space having a countable base and holds the property that each of its points has a neighbourhood homeomorphic either to an open subset of E^{n-1} or to a halfspace E^{n-1}_+. We note that this definition of topological manifold (see e.g. [128]) in our paper may be applied well. A relative boundary point of an $(n-1)$-manifold, lies on a bounding $(n-2)$-manifold of the original one. We note that the concept of boundary point of such a manifold is a topological invariant and a set homeomorphic to an $(n-1)$-dimensional hyperplane is a topological manifold without boundary points.

PROOF. (Lemma 2.1.3) Consider the boundary $\mathrm{bd}_{E^n} \mathrm{cl}_{E^n} H_{x,x}$ of closed set $\mathrm{cl}_{E^n} H_{x,x}$ (relative to the topology of E^n). (In general this set is a proper subset of $\mathrm{bd}_{E^n} H_{x,x}$.) By the assumption for y we see that this set does not contain y meaning that H_x contains an $(n-1)$-dimensional (separation) set $\mathrm{bd}_{E^n} \mathrm{cl}_{E^n} H_{x,x}$ and at least one maximal segment s' does not belong to this set. Since the set $\mathrm{bd}_{E^n} \mathrm{cl}_{E^n} H_{x,x}$ is closed there is a maximal non-degenerated subsegment s'' of s' (without right endpoint) which disjoint from $\mathrm{bd}_{E^n} \mathrm{cl}_{E^n} H_{x,x}$. If the point z is in $H_x \cap G'$ where G' is a smaller as G closed ball with center y then it has the same property as y, namely it has also a non-trivial segment in $H_x \setminus \mathrm{cl}_{E^n} H_{x,x}$. All of the segments parallel to x connecting the points of $H_x \cap G'$ with a corresponding point of $\mathrm{bd}_{E^n} \mathrm{cl}_{E^n} H_{x,x}$ determine a cylinder C with generator segments parallel to x. Of course the point y is an endpoint of a generator of this cylinder. Assuming now that H_x is a topological hyperplane C is a topological manifold, too. Thus C is a topological cylinder of dimension $(n-1)$. If now G'' is a smaller open ball as G' with center y then $G'' \cap H_x = G'' \cap C$ proving that y is a relative boundary point of H_x. This is a contradiction because the relative boundary of a topological hyperplane is empty. □

We have the following theorem:

THEOREM 2.1.2 ([1]). *If the unit ball K of a Minkowski normed space is strictly convex then all bisectors are homeomorphic to a hyperplane.*

PROOF. Since the Minkowski metric is invariant under translations we have to prove that if K is strictly convex then all sets H_x are homeomorphic image of a hyperplane.
Assume that the unit ball K is strictly convex. Let x be an arbitrary point of the space. Since K does not contain a segment on its boundary, from Lemma 2.1.2 we obtain that the intersections of H_x with every lines parallel to x contain exactly one point. Let now H be the $(n-1)$-dimensional subspace of E^n orthogonal to x and incident to the origin O and F be a map from this hyperplane H to H_x by x-projection with the definition:

$$F : H \longrightarrow H_x, y \longrightarrow F(y) = H_x \cap \{y + tx | t \in \mathbb{R}\}.$$

From Lemma 2.1.1 it follows that F is a bijective mapping from H to H_x we have to prove only that it is continuous one, with respect to the Euclidean metric topology. (The continuity of the inverse map will be a consequence of the fact that H is locally compact set.) Let now y be any point of H and $\epsilon > 0$ be arbitrary real number. Let z be a point of H for which the line $z + tx$ intersects the boundary of the K-ball K_1 with center 0 and radius $\|F(y)\|_K$. We have two parameters say t_1 and t_2 for which

$$\|z + t_1 x\|_K = \|F(y)\|_K \text{ and } \|z + t_2 x - x\|_K = \|F(y) - x\|_K = \|F(y)\|_K.$$

Since K convex compact body, the function from $H = \mathbb{R}^{(n-1)}$ to $\mathbb{R}$ giving those half of the boundary of K which contains the point $F(y)$ (with respect to an orthonormal base containing a unit vector parallel to x) is continuous. This means that we can choose a number $\delta > 0$ that if the Euclidean distance of z and y is less than δ then the distances of the points $z+t_1x, z+t_2x, F(y)$ are less than ϵ, respectively. Since the points $z + t_1x, z + t_2x$ belong to $H_{x,0}$ and $H_{x,x}$ or $H_{x,x}$ and $H_{x,0}$, respectively, we get that the corresponding segment $[z + t_1x, z + t_2x]$ contains the point $F(z)$. So the Euclidean distance of the image points $F(z)$ and $F(y)$ is also less than ϵ, meaning that F is continuous, so it is a homeomorphism. This proves the theorem. □

Illustrating the difficulties of the reversal problem now we consider three important examples.

EXAMPLE 2.1.1. Let the unit ball K be the cylinder defined by

$$K = \{(x, y, z) \in E^3 | -1 \leq x \leq 1, \quad y^2 + z^2 \leq 1\}.$$

The Leibnizian halfspaces of the vector $(2, 0, 0)$ are truncated open convex cones

$$\{(x, y, z) \in E^3 | x < 1, \quad 2 - x > \sqrt{y^2 + z^2}\} \text{ and } \{(x, y, z) \in E^3 | x > 1, \quad x > \sqrt{y^2 + z^2}\},$$

respectively. The topological dimension of H_x is three showing that it is not homeomorphic to a 2-plane. □

EXAMPLE 2.1.2. A more interesting fact that the unit sphere defined by the compact surface

$$r(t, s) := (2 - s^2)\cos(t)\mathbf{e}_1 + (1 - s^2)\sin(t)\mathbf{e}_2 + s\mathbf{e}_3, \text{ where } -1 \leq s \leq 1, \text{ and } 0 \leq t < 2\pi,$$

contains exactly two (opposite) segments with parameter values $s = \pm 1$. The bisector H_x of the vector $x = 4\mathbf{e}_1$ is the union of the plane $x = 2$ and the angular domains defined by the inequalities $\{y = 0, \quad x - 4 \geq z \geq x\}$ and $\{y = 0, \quad -x + 4 \leq z \leq -x\}$, respectively. This means that H_x belongs to two orthogonal planes of the space. For the proof that this set is not homeomorphic to a plane we have to see only that a set which is the union of two open circular disk with a common diameter can not be embedded topologically into a plane. In this topological space the separation theorem of Jordan does not hold because a closed Jordan curve in the plane of the first disk intersecting in two points of the common diameter, does not separate the all space. Hence this space is not homeomorphic an Euclidean plane as we stated. □

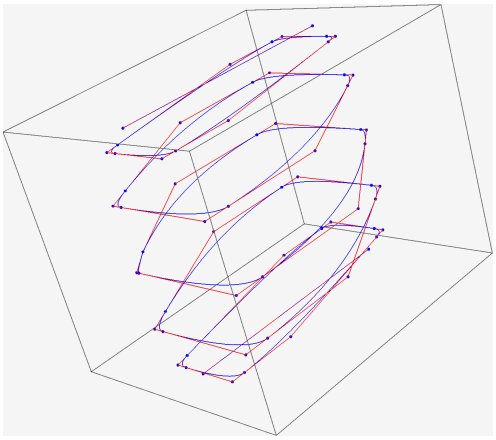

FIGURE 2.3. Six section splines of the unit ball K.

From this two examples it can be thought that if all bisectors are topological hyperplanes then K is strictly convex. The following example shows that it is not true in general.

EXAMPLE 2.1.3. K is an O-symmetric convex body of the three dimensional space bounded by the compact surface $r(u,v)$ defined by the following manner. Let $\gamma_u(v)$ be a closed parabolic Bezier spline containing the parabola segments determined by the points $P_i(u)$ $P_{i+2}(u)$ and the corresponding tangent lines $P_i(u)P_{i+1}(u)$ and $P_{i+1}(u)P_{i+2}(u)$, respectively, where $i = 0, 2, 4, 6, 8, 10$; $P_0(u) = P_{12}(u)$; $P_{6+i}(u) = -P_i(u) + [0, 0, 2\sin u]^T$ and the coordinates of the first six $P_i(u)$'s are

$$P_0(u) = \begin{bmatrix} 1+\varepsilon\cos u \\ 0 \\ \sin u \end{bmatrix} P_1(u) = \begin{bmatrix} 1+\varepsilon\cos u \\ \cos u \\ \sin u \end{bmatrix} P_2(u) = \begin{bmatrix} 1 \\ \cos u \\ \sin u \end{bmatrix} P_3(u) = \begin{bmatrix} 1-\cos u \\ \cos u \\ \sin u \end{bmatrix}$$

$$P_4(u) = \begin{bmatrix} -1 \\ \varepsilon\cos u \\ \sin u \end{bmatrix} P_5(u) = \begin{bmatrix} -1-\varepsilon\cos u \\ \varepsilon\cos u \cdot \frac{2-(2-\varepsilon)\cos u}{2-\cos u} \\ \sin u \end{bmatrix} P_6(u) = \begin{bmatrix} -1-\varepsilon\cos u \\ 0 \\ \sin u \end{bmatrix},$$

respectively. In Fig. 2.3 we can see the basic points $P_i(u)$ $(i = 0, \dots 6)$ and the corresponding splines for the parameter values $u = 0, \frac{\pi}{3}$ and $\frac{\pi}{2}$, and $\varepsilon = 0.25$, respectively.
Here ε is a non-negative constant (less or equal to $\frac{1}{2}$) $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$ is fixed and the parameter of $\gamma_u(v)$ is v, mapping the interval $[0, 6)$ onto the points of $\gamma_u(v)$. (The interval $[0, 1]$ mapped on the first parabola segment the interval $[1, 2]$ on the second one, etc.) Obviously $-\gamma_u(v) = \gamma_{-u}(3+v)$. The boundary of K is defined by the surface

$$r(u,v) := \{\gamma_u(v) | -\frac{\pi}{2} \le u \le \frac{\pi}{2}, 0 \le v < 6\}.$$

K is centrally symmetric convex body[1] with origin O for every $0 \le \varepsilon \le \frac{1}{2}$. If ε is positive that it is smooth and contains precisely two opposite segments at the parameter values $u = \pm\frac{\pi}{2}$. From the proof of the previous theorem we see that if the direction of x is not $[1, 0, 0]^T$ then H_x homeomorphic to a hyperplane. If now $x = (2 + 2\varepsilon, 0, 0)^T$ then H_x also homeomorphic to a hyperplane, though it contains two 2-dimensional angular domains of the plane $y = 0$. To prove this fact we note that the intersection of the two enlarged copies λK and $\lambda K + (2 + 2\varepsilon, 0, 0)^T$ in the case when $\lambda \ge 1 + \varepsilon$ is a closed Jordan curve, containing the parallel segments $s_1 = [(\lambda, 0, \lambda)^T, (2 + 2\varepsilon - \lambda, 0, \lambda)^T]$, $s_2 = [(\lambda, 0, -\lambda)^T, (2 + 2\varepsilon - \lambda, 0, -\lambda)^T]$ and two opposite (with respect to the center $P_0(0)$) curves connecting the point pairs $\{(\lambda, 0, \lambda)^T, (\lambda, 0, -\lambda)^T\}$, $\{(2 + 2\varepsilon - \lambda, 0, \lambda)^T, (2 + 2\varepsilon - \lambda, 0, -\lambda)^T\}$ where these curves are in the opposite space quarters $\{x \ge 1+\varepsilon, y \ge 0\}$, $\{x \le 1+\varepsilon, y \le 0\}$, respectively and if $1 \le \lambda \le 1+\varepsilon$ holds then this opposite parallel segments degenerate a point pair of the vertical segment $[(1, 0, 1+\varepsilon)^T, (1, 0, -1-\varepsilon)^T]$. Illustrating this situation we can figure of the most simple case when the parabola segments defined by the point pairs P_2P_4 and P_8P_{10} substituted by the line segments P_2P_4 and P_8P_{10}, respectively and $\varepsilon = 0$ and so the boundary of K is a ruled surface defined by two opposite closed half-circle). (See in Fig. 2.4) □

[1]The Gauss representation of the surface can be written concretely. The convexity can be checked from it using the fact that the support planes of the body does not intersects the interior of the body.

This example shows that a bisector H_x is homeomorphic to a hyperplane can contain $(n-1)$-dimensional cylinder with generators parallel to x implying the existence of a precisely $n-2$-dimensional cylinder on the boundary of K. We now formulate this observation in the following theorem.

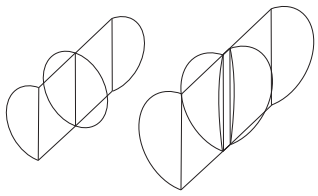

FIGURE 2.4. Two intersection curves in the case when $\varepsilon = 0$.

THEOREM 2.1.3 ([1]). *Let n be greater then two. If each of the bisectors is a topological hyperplane, then there is no $(n-1)$-dimensional cylinder on the boundary of K. Furthermore if H_x is a topological hyperplane and C is a maximal cylinder with generators parallel to x lying on $\operatorname{bd} K$ then it has dimension $(n-2)$.*

PROOF. The first statement of the theorem can be proved easily from the fact that every segment on the boundary induce an angular domain in the bisector H_x as we saw in the proof of Lemma 2.1.2. Hence If the boundary of K contains an $(n-1)$-dimensional cylinder then H_x contains an n-dimensional one.
We now prove the second statement of the theorem. Let C be any maximal cylinder of $\operatorname{bd} K$ with generators parallel to x. This means that the boundary of K in the direction of x contains C but there is no cylinder C' with the same direction of generators containing C and belonging also to $\operatorname{bd} K$ having greater dimension as of C. Let this dimension be k. C now induces a $(k+1)$-dimensional cylinder C^* with generators parallel to x in H_x containing maximal segments of H_x with the same direction. In Lemma 2.1.3 we showed that if H_x is topological hyperplane then all left end of every maximal segments of H_x containing the closed set $\operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,x}$. Obviously, the analogous statement is true for a right end of a maximal segment in H_x, meaning that it is in $\operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,0}$. Thus we have that in this case

$$H_x = \operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,0} = \operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,x}.$$

(The left ends and right ends of maximal segments evidently belong to $\operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,0}$ and $\operatorname{bd}_{E^n} \operatorname{cl}_{E^n} H_{x,x}$, respectively, and these two sets are also convex in the direction of x as H_x.) Let G be an n-dimensional ball with the radius ε. The points of $G + C^*$ can be divided into three sets S_0, S and S_x of $H_{x,0}$, H_x and $H_{x,x}$, respectively. Since the n-dimensional cylinder $G + C^*$ separated by S the dimension of S is at least $(n-1)$. Since $C^* \subset S$ we have two possibilities. In the first one S is a cylinder in H_x containing C^* and having greater dimension as of C^* while in the second case the two dimension is equal. The first possibility implies a cylinder in the boundary of K containing C and having dimension greater then of C. This contradicts to the assumption gave for C so the dimension of C^* is greater or equal to $(n-1)$. Thus the dimension of C is greater or equal to $(n-2)$. From the first note of this proof we can preclude the possibility of that this dimension is $(n-1)$ proving the second statement of the theorem. □

2.1.2. Dirichlet-Voronoi cells. We now turn out the problem of Dirichlet-Voronoi cells on the base of a K-ball above. First of all consider the following interesting example:

EXAMPLE 2.1.4. Let the unit ball is the square $[-1,1]^2$ of the plane and consider the lattice generated by the orthogonal vectors $(2,0)$ and $(0,16)$ (see Fig. 2.5.) The interior (open) Dirichlet-Voronoi cell of the point $(0,0)$ is the open convex hexagon bounded by the lines $x = \pm 1, y = \pm x \pm 2$, respectively. The exterior (closed) Dirichlet-Voronoi cell of the origin is the closure of the union of the interior Dirichlet-Voronoi cell and two concave pentagon with vertices $\{(0,2),(1,1),(8,8),(-8,8),(-1,1)\}$ and $\{(0,-2),(1,-1),(8,-8),(-8,-8),(-1,-1),$ respectively. The "wall" of this cell is a 2-dimensional subset of the plane. In this terminology the result of H.Mann says that if for all lattices of the space the exterior Dirichlet-Voronoi cells with respect to the considered Minkowski norm are convex then the unit ball of the norm is an ellipsoid. □

Because it is possible that the exterior Dirichlet-Voronoi cell is the Euclidean closure of the interior cell and they are not convex, we introduced the normality of the subdivision of the space generated by a lattice with respect to the examined Minkowski norm.

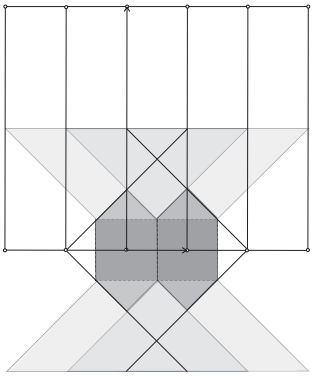

FIGURE 2.5. Open and closed Dirichlet-Voronoi cells of a lattice

DEFINITION 2.1.2 ([1]). *The Dirichlet-Voronoi cell system of a lattice L gives a* normal subdivision *of the embedding Euclidean space if the boundary of the cells does not contain n-balls.*

The following theorem gives a necessary and sufficient condition that all of the subdivisions being normal in the space.

THEOREM 2.1.4 ([1]). *The Dirichlet-Voronoi cell system of an arbitrary lattice L gives a normal subdivision of the embedding Euclidean space if and only if all bisectors are topological hyperplanes. Especially if the unit ball of the Minkowski norm is strictly convex then a lattice-like Dirichlet-Voronoi K-subdivision of any point lattice is normal.*

PROOF. If in the space there is a lattice which Dirichlet-Voronoi cell does not give a normal subdivision then there is n-dimensional ball belonging to the boundary of a cell. This means that there is a bisector which contains an n-dimensional ball.
Conversely, if all lattice-like Dirichlet-Voronoi cell subdivision are normal then all bisector is a topological hyperplane. In fact, if H_x is bisector G is an arbitrary open Euclidean ball with radius r and center $\frac{1}{2}x$, there is a lattice L for which the common wall of the Dirichlet-Voronoi cells of the origin and x (which are lattice points) contains the set $H_x \cap G$. (It is enough to choose a brick lattice generated by x and certain large vectors from its orthogonal complement.) Using normality and the fact that the exterior Dirichlet-Voronoi cell is a topological ball we get that this part of H_x is an elementary hypersurface. If now the radius r tends to infinity the statement is given.
Now the theorem follows from Theorem 2.1.3. □

2.1.3. On the shadow boundary of the unit ball in three-space. We examined in [2] the connections between the shadow boundaries of the unit ball K and the bisectors of the Minkowski space. Our conjecture is

CONJECTURE 2.1.1 ([2]). *The bisectors are topological hyperplanes if and only if the corresponding shadow boundaries are $(n-2)$-dimensional topological spheres.*

In [2] we proved this conjecture in the three-dimensional case. We examined also the topological properties of the shadow boundary, and defined the so-called *general parameter spheres* for $n \geq 3$, as a tool for a prospective proof of our conjecture.

DEFINITION 2.1.3. *Let K be a compact convex body in n-dimensional Euclidean space E^n and let S^{n-1} denote the $(n-1)$-dimensional unit sphere in E^n. For $x \in S^{n-1}$ the* shadow boundary *$S(K,x)$ of K in direction x consists of all points P in $\operatorname{bd} K$ such that the line $\{P+\lambda x : \lambda \in \mathbb{R}\}$ supports K, i.e. it meets K but not the interior of K. The shadow boundary $S(K,x)$ is* sharp *if any above supporting line of K intersects K exactly in the point P. If $S(K,x)$ is not sharp, in general, it may have* sharp point *for that the above uniqueness holds.*

It is clear that the shadow boundary decomposes the boundary of K into three disjoint sets. These are $S(K,x)$ itself, moreover

$$K^+ := \{y \in \operatorname{bd} K | \text{ there is } \tau > 0 \text{ such that } y - \tau \cdot x \in \operatorname{int}(K)\}, \tag{20}$$
$$K^- := \{y \in \operatorname{bd} K | \text{ there is } \tau > 0 \text{ such that } y + \tau \cdot x \in \operatorname{int}(K)\},$$

respectively. We call the congruent (thus homeomorphic) sets K^+ and K^- the *positive and negative part* of $\operatorname{bd} K$, respectively.

In general, the shadow boundary of a central symmetric convex body is not a nice set from topological point of view. There exists a central symmetric convex body K and a direction x of the space E^3 such that every supporting line of K parallel to x contains a point of K having no relative neighborhood in $S(K,x)$ homeomorphic to an open segment. This means that $S(K,x)$ is not a 1-dimensional manifold.

EXAMPLE 2.1.5. Consider a unit circle C in E^2 and the diadic rational points of it with respect to the usual parametrization (see in Fig. 2.6). More precisely, take the parameter values

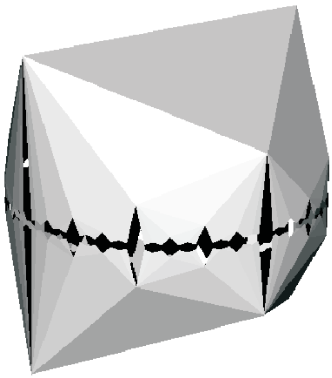

FIGURE 2.6. Shadow boundary which is not a topological manifold.

$t_{i,j} = \frac{j}{2^i}2\pi$, where $0 \leq i$ is integer and $1 \leq j \leq 2^i$ is odd number. The diadic rational points of the circle are the points $S_{i,j} = (\cos(t_{i,j}), sin(t_{i,j}))$ of the subspace E^2 with respect to an orthonormed basis. Let now $s_{i,j}$ be a segment orthogonal to the subspace E^2 whose midpoint is $S_{i,j}$ and its length is equal to $\frac{1}{2^{i-2}}$ if $i \geq 2$ and is equal to 2 if $i = 0, 1$. The point sets

$$C^* := C \cup (\cup_{i,j}\{s_{i,j}\}) \text{ and } K := \operatorname{conv} C^*$$

are central symmetric. This body is also closed, see it in Fig. 2.6. If l is a supporting line of K orthogonal to the plane E^2 then it does not intersect the relative interior of the disc in E^2 bounded by the circle C, so it intersects the circle C. If $l \cap C$ is a point of form $S_{i,j}$ then $l \cap K = l \cap C^* = s_{i,j}$, while if $l \cap C$ is another point of C then $l \cap K = l \cap C$. We conclude to $S(K,x) = C^*$ being not a 1-manifold, as we claimed. □

In order to describe the connection between the bisectors and the shadow boundaries of the unit ball we introduce some parameterized sets on the boundary of K, corresponding to a given direction of the space. These tend to the shadow boundary of K of the same direction if the parameter tends to infinity. As we shall see in the case of a nice unit ball these sets give a parametrization of the closed "positive part" of $\mathrm{bd}K$. In this way we can define the *general parameter spheres* according to this direction.

DEFINITION 2.1.4 ([2]). *Let K be the Minkowski unit ball above and x is a fixed direction of the space E^n. Let*

$$\lambda_0 := \inf\{0 < t \in \mathbb{R} \mid tK \cap (tK + x) \neq \emptyset\}$$

be the smallest value t for which tK and $tK + x$ intersect. Then a general parameter sphere of $\mathrm{bd}K$ *corresponding to the direction x and to any fixed parameter $\lambda \geq \lambda_0$ is the following set:*

$$\gamma_\lambda(K,x) := \frac{1}{\lambda}\left(\operatorname{bd}(\lambda K) \cap \operatorname{bd}(\lambda K + x)\right) \subset \operatorname{bd} K.$$

In general, the above set is not a topological sphere of dimension $(n-2)$, and they are not homeomorphic to each other for different λ's. For example the dimension of $\gamma_{\lambda_0}(K,x)$ may be $0, 1 \cdots (n-1)$ while the dimension of $\gamma_\lambda(K,x)$ for $\lambda > \lambda_0$ is at least $(n-2)$ because it dissects the boundary of K. We also remark that the two parts of $\operatorname{bd} K \setminus \gamma_\lambda(K,x)$ for $\lambda > \lambda_0$ are also homeomorphic to each other by the projection from $\frac{1}{2\lambda}x$ (since $\lambda K \cap \lambda K + x$ is central symmetric in $\frac{1}{2}x$ for any $\lambda \geq \lambda_0$).

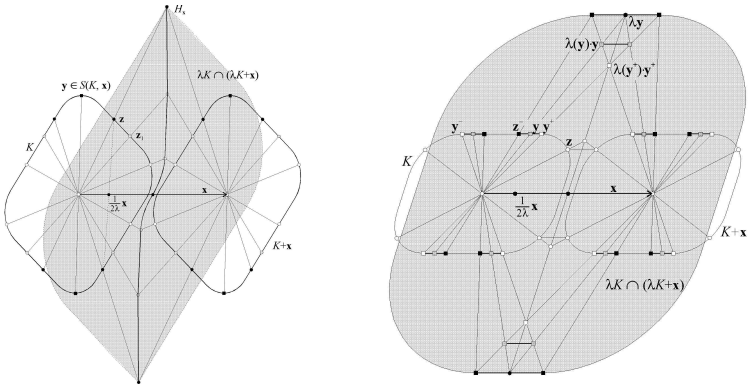

FIGURE 2.7. The shadow boundary could be sharp or not sharp in y

LEMMA 2.1.4 ([2]). *Let $\Pi(x,y)$ be a 2-plane parallel to the vectors x and $y \in S(K,x)$, through the origin. Then we have two possibilities for $\Pi(x,y) \cap \gamma_\lambda(K,x)$:*

- *If the shadow boundary $S(K,x)$ is sharp for the point $y \in S(K,x)$ then $\Pi(x,y) \cap \gamma_\lambda(K,x)$ contains two opposite points with respect to $\frac{1}{2\lambda}x$ (Fig.2.7 (left))*
- *There is a uniquely defined parameter value $\lambda(\mathbf{y})$ that for every $\lambda > \lambda(y)$ the intersection $\Pi(x,y) \cap \gamma_\lambda(K,x)$ is the union of a pair of segments parallel to x, opposite with respect to $\frac{1}{2\lambda}x$. (Fig.2.7 (right))*

In the second case the segments of the parameter spheres $\gamma_\lambda(K,x)$ belong to the shadow boundary $S(K,x)$.

PROOF. Let $\lambda > \lambda_0$ be an arbitrary real number and consider the generalized parameter sphere $\gamma_\lambda(K,x)$. Then $\gamma_\lambda(K,x) = \frac{1}{\lambda}S(\lambda K \cap (\lambda K + x), x)$. In fact, $y \in \gamma_\lambda(K,x)$ if and only if $\lambda y \in bd(\lambda K) \cap \mathrm{bd}(\lambda K + x) \subset \mathrm{bd}(\lambda K \cap (\lambda K + x))$. Let the line $l(\tau)$ be of the form $\lambda y + \tau x$ where τ runs through real numbers.
There is no $\tau_0 \neq 0$ for which e.g. $\tau_0 < 0$ holds and $\lambda y + \tau_0 x \in \mathrm{int}(\lambda K \cap (\lambda K + x))$. Indirectly, $\lambda y + \tau_0 x \in \mathrm{int}(\lambda K)$ and $\lambda y + \tau_0 x \in \mathrm{int}(\lambda K + x)) = \mathrm{int}(\lambda K) + x$ hold. The second relation implies $\lambda y + (\tau_0 - 1)x \in \mathrm{int}(\lambda K)$, while $\lambda y \in \mathrm{bd}(\lambda K)$ and $\lambda y \in \mathrm{bd}(\lambda K + x)$ involve $\lambda y - x \in \mathrm{bd}(\lambda K)$. This means that the points λy, $\lambda y - x$, $\lambda y + \tau_0 x$, $\lambda y + (\tau_0 - 1)x$ are on the line l, ordered as

$$\lambda y - x, \lambda y + (\tau_0 - 1)x, \lambda y + \tau_0 x, \lambda y$$

by the convexity of K. This would imply $\tau_0 = 0$, a contradiction.
Since the shadow boundary of the convex bodies $K_\lambda = \frac{1}{\lambda}(\lambda K \cap (\lambda K + x))$ to x are on the boundary of K, it can contain a segment parallel to x if and only if this segment belongs to the shadow boundary of K, too. An interesting phenomenon that – though $\Pi(x,y) \cap S(K,x)$ is a pair of opposite segments (by central symmetry in 0) – for a starting λ (which gives the positive end of $\Pi(x,y) \cap S(K,x)$), $\Pi(x,y) \cap \gamma_\lambda(K,x)$ is a pair of points. So we are done. □

An important consequence of Lemma 2.1.4 is the following

COROLLARY 2.1.1. The general parameter spheres for $\lambda > \lambda_0$ provide a natural parametrization of the surface $K^+ \setminus \gamma_{\lambda_0}(K,x)$. In this parametrization any point of $K^+ \setminus \gamma_{\lambda_0}(K,x)$ is determined by a point of a Euclidean unit sphere of dimension $(n-2)$, orthogonal to x in 0, and by a parameter $\lambda > \lambda_0$.

Of course, it is possible that the above surface $K^+ \setminus \gamma_{\lambda_0}(K,x)$ is empty, as in the case of a cube (=K) when four of its edges is parallel to x. However, in significant cases it is a useful parametrization. For example, if K is strictly convex, then it has only one singular point $\gamma_{\lambda_0}(K,x)$ on the positive half.
To prove this corollary, we observe the fact that the common points of two distinct parameter spheres belong to the shadow boundary of K, hence the generalized parameter spheres give a one-fold covering of $K^+ \setminus \gamma_{\lambda_0}(K,x)$.

We recall the concept of *Hausdorff distance* ρ_H of two point sets S_1 and S_2, expressed by the Euclidean distance ρ_E:

$$\rho_H(S_1, S_2) = \max\{\sup_{s_1 \in S_1}\{\rho_E(s_1, S_2)\}, \sup_{s_2 \in S_2}\{\rho_E(s_2, S_1)\}\}.$$

(Here e.g. $\rho_E(s_1, S_2) = \inf_{s_2 \in S_2}\{\rho_E(s_1, s_2)\}$.)
Our main result on general parameter spheres is the following:

THEOREM 2.1.5 ([2]). *The shadow boundary $S(K, x)$ is the limit of the general parameter spheres $\gamma_\lambda(K, x)$, with respect to the Hausdorff metric, when λ tends to infinity.*

PROOF. According to the previous lemma we have two cases. In the first one the 2-plane $\Pi(x, y)$, with $y \in S(K, x)$, intersects both $S(K, x)$ and $\gamma_\lambda(K, x)$ in two point pairs, respectively (Fig. 2.7 (left)); while in the second case the intersection $\Pi(x, y) \cap S(K, x)$ is a 0-opposite pair of segments, and the intersection $\Pi(x, y) \cap \gamma_\lambda(K, x)$, if $\lambda > \lambda(y) \geq \lambda_0$, is an opposite pair of segments with respect to $\frac{1}{2\lambda}x$ (Fig.2.7 (right)). We will mention the necessary intersections as a point or a segment, shortly. Let S' be the set of sharp points of $S(K, x)$ and S'' be the set of the remaining points of $S(K, x)$, decomposed to (disjoint) segments parallel to x. We say that the points $y \in S(K, x)$ and $z \in \gamma_\lambda(K, x)$ correspond to each other, if $y, z \in \Pi(x, y)$ and the line of direction x through the origin does not separate them in $\Pi(x, y)$. If $y \in S'$ then there exists one corresponding point $z \in \gamma_\lambda(K, x)$ (See Lemma 2.1.4). Denote this simply by z. If $y \in S''$ then either it has only one corresponding point in $\gamma_\lambda(K, x)$ (see Lemma 2.1.4, $\lambda_0 < \lambda \leq \lambda(y)$) or the corresponding points form a segment belonging to S'' (Lemma 2.1.4, $\lambda > \lambda(y)$). We focus on the negative end of the segment of S'', containing y denoted by y^-, and the negative end of the corresponding segment of $\gamma_\lambda(K, x)$ denoted by z^-. Let S''' be the set of those points z of $\gamma_\lambda(K, x)$ which correspond to a point of S', and S'''' be the collection of the remaining points of $\gamma_\lambda(K, x)$. Now the claimed convergence follows from the inequalities below:

$$\rho_H(S(K,x), \gamma_\lambda(K,x)) = \max\{\sup_{y \in S(K,x)}\{\rho_E(y, \gamma_\lambda(K,x))\}, \sup_{z \in \gamma_\lambda(K,x)}\{\rho_E(S(K,x), z)\}\} =$$

$$= \max\{\sup_{y \in S'}\{\rho_E(y, \gamma_\lambda(K,x))\}, \sup_{y \in S''}\{\rho_E(y, \gamma_\lambda(K,x))\}, \sup_{z \in S'''}\{\rho_E(S(K,x), z)\}, \sup_{z \in S''''}\{\rho_E(S(K,x), z)\}\} \leq$$

$$\leq \max\{\sup_{y \in S'}\{\rho_E(y, z)\}, \sup_{y^- \in S''}\{\rho_E(y^-, \gamma_\lambda(K,x))\}, \sup_{z^- \in S'''}\{\rho_E(y^-, z^-)\}, \sup_{z \in S''''}\{\rho_E(S(K,x), z)\}\} \leq$$

$$\leq \max\{\sup_{y \in S'}\{\rho_E(y, z)\}, \sup_{y^- \in S''}\{\rho_E(y^-, z^-)\}, \sup_{z \in S'''' \setminus S(K,x)}\{\rho_E(S(K,x), z)\}\} \leq$$

$$\leq \max\{\sup_{y \in S'}\{\rho_E(y, z)\}, \sup_{y^- \in S''}\{\rho_E(y^-, z^-)\}, \sup_{z \in S'''' \setminus S(K,x)}\{\rho_E(y^-, z)\}\}.$$

In fact, each of these three Euclidean distances tend to zero, if λ tends to infinity, since K and its two dimensional intersections are convex and compact, respectively. □

On the rest of this section we restrict the investigation to the case of dimension 3. A point set $H \subset E^3$ is said to be a *topological plane* if and only if there is a homeomorphism of E^3 onto itself, sending H onto a usual 2-plane. We recall a theorem of two-dimensional topology, characterizing the topological circles on a two-sphere. (See for example [145].) A point a is called *arcwise accessible* from a point set B if $b \in B$ implies the existence of an arc T with end points a and b such that $T \setminus a \subset B$. If A is a point set whose every point is arcwise accessible from some point set B, then we call A *arcwise accessible from* B. We use the Schoenflies-Swingle theorem:

THEOREM 2.1.6 (Schoenflies, Swingle see in [134] and [137]). *A necessary and sufficient condition that a subset M of S^2 should be an S^1 is that it be a common boundary of two disjoint domains D_1 and D_2, from which M is arcwise accessible.*

Now our first statement is a technical lemma.

LEMMA 2.1.5 ([2]). *Assume that the shadow boundary $S(K, x)$ contains a segment s parallel to x having the property that it is a subset of accumulation points of $S(K, x) \setminus s$. Then the bisector H_x can not be a topological plane.*

PROOF. Let y be a relative inner point of the segment s of accumulation points of $S(K,x)$. There exists such a λ (large enough) and also an ε (small enough) for which the segment with negative end y and positive end y^+ of s lies in $\gamma_{\lambda'}(K,x)$ where $\lambda-\varepsilon<\lambda'<\lambda+\varepsilon$ and the accumulation points of the sets $\gamma_{\lambda'}(K,x)\setminus s$ contain also the segment $[y,y^+]$. This means, there is a domain – namely the union of segments $\cup_{\lambda'}\{\lambda'[y,y^+]|\lambda-\varepsilon<\lambda'<\lambda+\varepsilon\}$ – in the bisector H_x which lies in the set of accumulation points of the complementary set with respect to H_x. Drawing in this domain a little circle we get a closed curve which relative interior points are also boundary points of its complementary sets. Thus the Jordan Curve Theorem (as a special case of the Schoenflies-Swingle theorem) does not hold on H_x, consequently H_x could not be a topological plane. □

THEOREM 2.1.7 ([2]). *Assume that the bisector H_x is a topological plane of E^3. Then the general parameter spheres $\gamma_\lambda(K,x)$ for $\lambda>\lambda_0$ and the shadow boundary $S(K,x)$ are topological 1-manifolds (topological circles). For $\lambda=\lambda_0$ the parameter sphere can form a point, a segment or a convex disk of dimension 2, respectively.*

PROOF. Firstly, we deal with general parameter spheres. The statement on $\gamma_{\lambda_0}(K,x)$ follows from the convexity and central symmetry of the compact body K (and $K+x$ as well).
For $\lambda>\lambda_0$ we prove that $\lambda(\gamma_\lambda(K,x))\subset H_x$ is arcwise accessible from the negative sets

$$H_1'=\cup_{\lambda'}\{\lambda'(\gamma_{\lambda'}(K,x))|\lambda_0\le\lambda'<\lambda\}\subset H_x\subset H_x^-,$$

If v is a point of $\lambda(\gamma_\lambda(K,x))$ then there is an arc, parameterized by λ' in the intersection $H_x\cap\Pi(x,v)$ which connect the point v with the point $\frac{1}{2}x$, with the property that their points, different from v, lie in H_1'. Since also $\lambda\gamma_\lambda(K,x)$ is the common boundary of H_1' and its complementary set in H_x, by the Schoenflies-Swingle theorem, we get that $\lambda\gamma_\lambda(K,x))$ is a topological circle, i.e. by the projection from 0, $\gamma_\lambda(K,x))$ is a topological circle, too, which is arcwise accessible also from the open disk component of $\mathrm{int}(K^+\setminus\gamma_\lambda(K,x)$ by Theorem 2.1.6.
Now let's turn to the case of the shadow boundary: We assume that H_x is a topological plane. We check that the conditions of Schoenflies-Swingle theorem hold for $S(K,x)$, too. It is enough to prove that $S(K,x)$ is arcwise accessible from K^+. Let y an arbitrary point of $S(K,x)$.
If $S(K,x)$ is sharp at this point then, by Lemma 2.1.4 , the set

$$\cup_\lambda\{\Pi(x,y)\cap\gamma_\lambda(K,x)|\lambda\ge\lambda_0\}\cup y$$

is a good arc which connects the interior of K^+ and y. (Since K^+ is arcwise connected y is accessible from points K^+ by arcs.)
If y is not a sharp point of $S(K,x)$ then (by Lemma 2.1.4) we have the segment s of $S(K,x)$ through y as a union of the monotone increasing sequence of segments $\Pi(x,y)\cap\gamma_\lambda(K,x)$, parallel to x where $\lambda>\lambda(y)$, and the negative end y^- of s (Fig. 2.8).
Observe that all of this segments are arcwise accessible from K^+, so is their union, too. To prove this, let s' denote one of the segments $\Pi(x,y)\cap\gamma_\lambda(K,x)$ for fixed $\lambda>\lambda(y)$. Observe that the points of K^+ belong to one of the following three sets:

$$H_1=\cup_{\lambda'}\{\gamma_{\lambda'}(K,x)\mid\lambda>\lambda'\ge\lambda_0\}\cap K^+,\gamma_\lambda(K,x)\cap K^+\text{ and }K^+\setminus(\gamma_\lambda(K,x)\cup H_1).$$

From the points of the first set (by the first part of this proof) there are arcs connecting a point y' of the considered segment with the required property. We can connect the points of the second set with a point of H_1 by such an arc whose points belong to K^+, and this latter point can be connected again with a required arc, showing that from these points there also exist arcs to y'. Finally, a point v of the third set lies in a plane $\Pi(x,v)$ intersecting $S(K,x)$ in a sharp point. The arc from v to a point of H_1 in the intersection $\Pi(x,v)\cap\mathrm{bd}\,K$ can be extended to a required arc with ends at y'.
It remains to examine of the negative end point y^- of s (see Fig.2.8). Since y^- is a boundary point of the segment s whose other points belong to the boundary of K^+, then it is a boundary point of K^+. Consider now a sequence (z_i) of points of K^+ that tends to y^-. First we introduce a parametrization of $S(K,x)\cup K^+$. Let (φ,ψ) denote the coordinates of any point $z\in\mathrm{bd}\,K$.

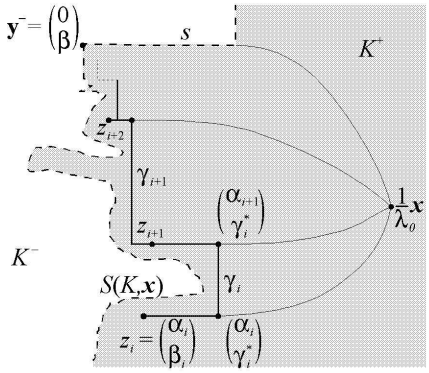

FIGURE 2.8. The negative end is accessible by arc.

Here φ is the angle of the planes $\Pi(x,z)$ and $\Pi(x,y^-)$ $-\pi < \varphi \leq \pi$ with respect to a fixed orientation, and ψ the angle of the vectors x and z, $0 < \psi < \pi$. Then we have $(z_i) = ((\alpha_i, \beta_i)^T)$ and $y^- = (0,\beta)^T$, T means transposed. We can assume, without loss of generality, that the sequence (α_i) is monotone decreasing. Now we connect the points z_i and z_{i+1} by an arc γ_i lying in K^+. We define ψ_i^* for later arcs, near enough $S(K,x)$, by

$$\psi_i^* := \inf\{\psi| \text{ there exists } \alpha_i \geq \varphi \geq \alpha_{i+1} \text{ for which } (\varphi,\psi)^T \in S(K,x)\} - \frac{1}{2^i}.$$

From now on the notation $x \in [a,b]$ $(x \in (a,b))$ means that either $a \leq x \leq b$ $(a < x < b)$ or $a \geq x \geq b$ $(a > x > b)$ hold. Then the arc γ_i connecting z_i and z_{i+1} is the following:

$$\gamma_i := \{(\alpha_i,\psi)^T \text{ with parameter } \psi \in [\beta_i,\psi_i^*]\} \cup \{(\varphi,\psi_i^*)^T \text{ with } \varphi \in (\alpha_i,\alpha_{i+1})\} \cup$$
$$\cup\{(\alpha_{i+1},\psi)^T \text{ with } \psi \in [\beta_{i+1},\psi_i^*]\}.$$

Of course, the simple union of these arcs is considered only one curve for which one of its accumulation points is $y^- = (0,\beta)^T$. However, the following set $\gamma := cl(\cup_i\gamma_i \setminus \cup_i(\gamma_i \cap \gamma_{i+1}))$ (in which we do not take multiple points) is an appropriate arc if and only if $\gamma \setminus \cup_i\gamma_i = \{y^-\}$. Since the set of accumulation points of γ is a subset of $\gamma \cup s$, thus the indirect assumption implies a subsegment s' of s with non-zero length. This is also a subset of accumulation points of $S(K,x) \setminus s$ and applying the Lemma 2.1.5 we get that the bisector would not be a topological plane. Thus the conditions of the Schoenflies-Swingle theorem are fulfilled so $S(K,x)$ is a topological circle as we claimed. □

LEMMA 2.1.6 ([2]). *Assume that the shadow boundary of K in the direction x is a topological circle. Then the general parameter spheres are also topological circles for $\lambda > \lambda_0$.*

The proof is an easy consequence of Theorem 2.1.6 and of the arguments before it. The main result of this section is:

THEOREM 2.1.8 ([2]). *Let K be a central symmetric compact convex body in E^3. All of the bisectors H_x of the corresponding Minkowski normed space are topological planes if and only if all of the shadow boundaries $S(K,x)$ are topological circles (1-spheres).*

PROOF. The necessity is a consequence of Theorem 2.1.7.
We prove that if the shadow boundary is a topological circle then the corresponding bisector H_x is a topological plane. By the assumption and Lemma 2.1.6, $\gamma_\lambda(K,x)$ is a topological circle for any fixed $\lambda > \lambda_0$, and $\gamma_{\lambda_0}(K,x)$ is a topological closed ball of dimension 0,1 or 2, respectively. Consider now $S(K,x)$.
First we note that, for a fixed λ, on $\gamma_\lambda(K,x)$ there are only finitely many segments parallel to x. In the contrary case there would be infinitely many corresponding segments on $S(K,x)$, too, but $S(K,x)$ is compact and homeomorphic to a circle, this would easily lead to a contradiction with Theorem 2.1.6. Then the set of lengthes of these segments of $S(K,x)$ has a positive lower bound. Thus there are only finitely many parameter values λ_i with the property that $\gamma_{\lambda_i}(K,x)$

$(\lambda_i > \lambda_0)$ contains such a positive end of a segment s_i of the shadow boundary parallel to x, which is not lying on a $\gamma_{\lambda'}(K,x)$ for $\lambda' < \lambda_i$.
If y_i^+ is a positive end of s_i then $\lambda_i y_i^+$ is an apex of a corner domain belonging to the intersection of H_x and a plane through the origin and s_i. Partition now H_x into non-overlapping rings by the consecutive topological circles $\lambda_i\gamma_{\lambda_i}(K,x)$ $i \geq 1$. A ring between the circles $\lambda_i\gamma_{\lambda_i}(K,x)$ and $\lambda_{i+1}\gamma_{\lambda_{i+1}}(K,x)$ can be partitioned by straight-line boundaries of the corresponding corners to finitely many non-overlapping domains $D_{i,j}$ where $D_{i,j} \cap D_{i,j+1}$ (for every j with respect to a cyclic order, is a segment connecting a point of $\lambda_i\gamma_{\lambda_i}(K,x)$ to a point of $\lambda_{i+1}\gamma_{\lambda_{i+1}}(K,x)$. These closed domains (each homeomorphic to a closed disc for $i \geq 1$) join only finitely many others, thus we can define a sequence of homeomorphisms $\Phi_{i,j}$ on $D_{i,j}$ by induction in the following way.
First, we partition the unit disc B (with center O) into non-overlapping pieces having the same combinatorial structure as the subdivision of $H_x = \lambda_0\gamma_{\lambda_0}(K,x) \cup_{i,j} D_{i,j}$. We have three cases: $\lambda_0\gamma_{\lambda_0}(K,x)$ is a closed disc, a closed segment or a point.
In the first case we consider the concentric circles C_{λ_i} with respective radii $r_{\lambda_i} = 1 - \frac{\lambda_0}{2\lambda_i}$ for $i \geq 1$ and define the image of $\lambda_0\gamma_{\lambda_0}(K,x)$ as the disk with origin O and radius $\frac{1}{2}$.
In the second case we consider concentric ellipses which converges to a O-symmetric segment of length 1, and the third case the ring structure giving by concentric circles, too, with corresponding radii $r_{\lambda_i} = 1 - \frac{\lambda_0}{\lambda_i}$ for $i \geq 1$.
We map now the shadow boundary $S(K,x)$ onto the boundary of B. A corner domain of H_x corresponds to a segment s of $S(K,x)$ thus also to a closed arc σ of the unit circle. On the other hand the apex a_σ of this corner corresponds to a λ_i. If $i > 0$ let a'_σ a point of $C_{\lambda_i} \cap \mathrm{conv}\{O,\sigma\}$. For $i = 0$, in the first case, we may choose a'_σ in the same way; in the second case we have only two possibilities for a_σ (the ends of $\lambda_0\gamma_{\lambda_0}(K,x)$); thus let a'_σ be one of the ends of the corresponding segment C_{λ_0}. (In this case we choose the corresponding arc γ_0 intersecting the line of C_{λ_0}. Finally in the latter case there is no such apex. Now we subdivide the rings by the sectors $\mathrm{conv}\{a'_\sigma,\sigma\}$. Obviously, the domains $Q_{i,j}$ in this process can be corresponded to the domains $D_{i,j}$ in a unique way. This means that we decomposed B to closed domains $Q_{i,j}$ with the property: $\cap D_{i,j}$ is homeomorphic to $\cap Q_{i,j}$ for indices i,j.
Second, by induction (with respect to the lexicographic order of the pairs (i,j)) it is not to hard to give a family $\{\Phi_{i,j} : D_{i,j} \longrightarrow Q_{i,j}\}$ of homeomorphisms compatible to each other, requiring that if $D_{i,j} \cap D_{k,l} \neq \emptyset$ then $\Phi_{i,j}(v) = \Phi_{k,l}(v)$ for each point v of $D_{i,j} \cap D_{k,l}$. (Denote by $\Phi_{0,0}$ the first homeomorphism sending $\lambda_0\gamma_{\lambda_0}(K,x)$ onto the corresponding (not-indicated) subset of B.) Now the mapping $\Phi : H_x \longrightarrow \mathrm{int}\, B$ (see Fig 2.9), sending a point $v \in D_{i,j}$ to the point $\Phi_{i,j}(\mathbf{v})$, is evidently a homeomorphism of H_x onto the interior of the disc B as we stated. □

2.1.4. Bisector and shadow boundary in higher spaces. The examination of Conjecture 2.1.1 in higher dimension require a deeper investigation of the topological properties of the general parameter spheres. The corresponding results of the author can be found in the paper [**3**]. We proved that, the general parameter spheres are not an absolute neighborhood retract (ANR) in general, but still are compact metric spaces, containing $(n-2)$-dimensional closed, connected subsets separating the boundary of K. Thus we investigated the manifold case and we proved that the general parameter spheres and the corresponding shadow boundary are homeomorphic to the $(n-2)$-dimensional sphere. The base of the proof is the so-called *cell-like approximation theorem for manifolds*. The long history of it can be found for example in [**124**].

THEOREM 2.1.9 (Cell-like Approximation Theorem for manifolds). *Let $n \neq 3$ be a positive integer. For every cell-like map $f : M \longrightarrow N$ between topological n-manifolds, and every $\varepsilon > 0$, there is a homeomorphism $h : M \longrightarrow N$ such that $d(f,h) < \varepsilon$ in the sup-norm metric on the space of all continuous maps (so f is a so-called* near homeomorphism*).*

We use again the notation:

$$K^+ := \{y \in \operatorname{bd} K | \text{ there is } \tau > 0 \text{ such that } y - \tau \cdot x \in \operatorname{int}(K)\},$$
$$K^- := \{y \in \operatorname{bd} K | \text{ there is } \tau > 0 \text{ such that } y + \tau \cdot x \in \operatorname{int}(K)\}.$$

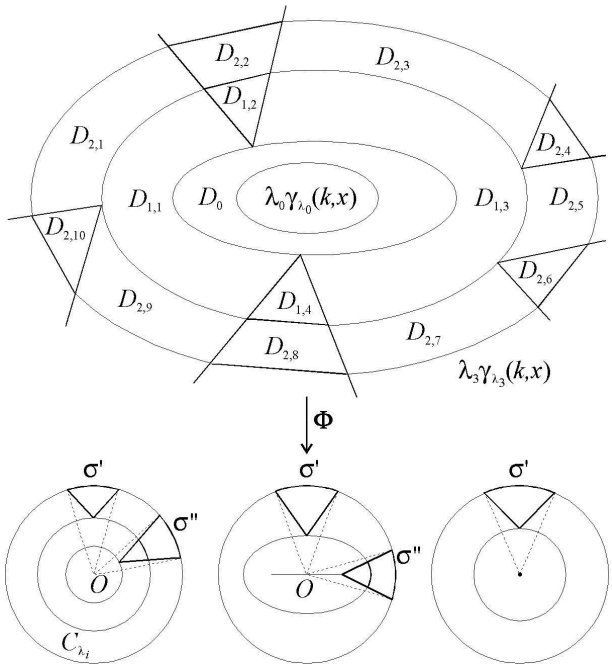

FIGURE 2.9. The homeomorphism Φ

We call the congruent (thus homeomorphic) sets K^+ and K^- the *positive and negative part* of $\operatorname{bd} K$, respectively. The line passing through the origin and parallel to the vector x intersects the boundary of K at the points $P^+ \in K^+$ and $P^- \in K^-$ showing that the positive and negative part of $\operatorname{bd} K$ are not empty, respectively. We call the points P^+ and P^- the *positive* and *negative pole* of K, respectively. The intersection of $\operatorname{bd}(K)$ by a 2-plane containing the poles is called a *longitudinal parameter curve* of K.

STATEMENT 2.1.1 ([3]). *The shadow boundary decomposes the boundary of K into three disjoint sets: $S(K,x)$, K^+ and K^-. $S(K,x)$ is an at least $(n-2)$-dimensional closed (so compact) set in $\operatorname{bd} K$ which is connected for $n \geq 3$, the sets K^+ and K^- are homeomorphic copies of $\mathbb{R}^{(n-1)}$ giving two arcwise connected components of their union.*

PROOF. The first statement is obvious. Let p_x be the orthogonal projection of the embedding space R^n onto a hyperplane orthogonal to the vector x. Since the orthogonal projection is a contraction then it is continuous mapping of the space. $p_x(K)$ is a convex body of the image hyperplane. The interior of $p_x(K)$ is the image of the sets K^+ and K^-, respectively and its boundary is the image of $S(K,x)$. Since p_x restricting for K^+ is a bijection, there exists a homeomorphism on K^+ to $\mathbb{R}^{(n-1)}$. Using the same argument for K^- we proved the validity of the first part of the statement on K^+ and K^-. Of course their union is open therefore the shadow boundary is closed.
Since $\mathbb{R}^{(n-1)}$ is arcwise connected the second part of the statement on K^+ follows from the fact that an arc connecting two points of K^+ and K^- should be decomposed into two relative open sets by K^+ and K^-, which is a contradiction. Thus the shadow boundary separates the boundary of K. By a theorem of Alexandrov (Th. 5.12 in vol.I of [16]), we get, that the topological dimension of $S(K,x)$ is at least $(n-2)$, as we stated.
We now prove that (for $n \geq 3$) the set $S(K,x)$ is connected. Assume that K_1 and K_2 are two closed disjoint subsets of the shadow boundary for which $K_1 \cup K_2 = S(K,x)$. First we observe that each of the metric segments lying on a longitudinal parameter curve and parallel to x is a connected subset of $S(K,x)$, thus its points (by the "basic lemma of connectivity" see vol.I p.13 in [16]) belong either to the set K_1 or to the set K_2. Let C_1 and C_2 the sets defined by the union of those longitudinal parameter curves which intersect the sets K_1 and K_2. In this case $C_1 \cup C_2 = \operatorname{bd} K$ and $C_1 \cap C_2 = \{P^+, P^-\}$ hold. The sets C_i are closed in $\operatorname{bd} K$, meaning that the sets $C_i \setminus \{P^+, P^-\}$ give a decomposition of $\operatorname{bd} K \setminus \{P^+, P^-\}$ into disjoint relative closed subsets, too. Since the latter set is connected it follows that either K_1 or K_2 is empty. □

In general the dimension of $S(K,x)$ is $(n-2)$ or $(n-1)$. We prove that there is an $(n-2)$-dimensional closed, connected subset of $S(K,x)$ separating $\operatorname{bd} K$, too.

LEMMA 2.1.7 ([3]). *The boundary of the closure of the set K^+ (denoted by $\operatorname{bd}(\operatorname{cl}(K^+))$) is a closed, connected $(n-2)$ dimensional subset of $S(K,x)$ separating the boundary of K.*

PROOF. By its definition it is closed. Since $\mathrm{cl}(K^+) \supset K^+$ and $\mathrm{cl}(K^+) \cap K^- = \emptyset$ we have $K^+ \subset \mathrm{cl}(K^+) \subset K^+ \cup S(K,x)$. On the other hand $\mathrm{bd}(\mathrm{cl}(K^+)) \cap K^+ = \emptyset$ (K^+ is an open subset of $\mathrm{cl}(K^+)$), thus we get that $\mathrm{bd}(\mathrm{cl}(K^+)) \subset S(K,x)$.
The separating property follows from the fact that the union of the pairwise disjoint sets $\mathrm{bd}\, K \setminus \mathrm{cl}(K^+)$, $\mathrm{int}(\mathrm{cl}(K^+))$, $\mathrm{bd}(\mathrm{cl}(K^+))$ fills the boundary of K and the first two sets are open. Now the separating property implies (again by the Alexandrov's theorem above) the inequality $\dim(\mathrm{bd}(\mathrm{cl}(K^+))) \geq (n-2)$. On the other hand a closed connected set of dimension $(n-1)$ on $\mathrm{bd}\, K$ contains an interior point relative to $\mathrm{bd}\, K$ (see p.174 in vol I. of [16]) which contradicts to the definition of $\mathrm{bd}(\mathrm{cl}(K^+))$. □

Now we can prove one of the main theorems of this dissertation.

THEOREM 2.1.10 ([3]). *If the shadow boundary $S(K,x)$ is a topological manifold of dimension $(n-2)$ then it is homeomorphic to the $(n-2)$-sphere $S^{(n-2)}$. If it is an $(n-1)$-dimensional manifold with boundary then it is homeomorphic to the cylinder $S^{(n-2)} \times [0,1]$.*

PROOF. Consider first the projection p_x (which was defined in the proof of Statement 2.1.1), and restrict it to the shadow boundary of K parallel to x. It is a cell-like map because of the inverse images are points or segments, respectively. In this way for $n \neq 5$ by the approximation theorem (Theorem 2.1.9) above we have that this restricted map is a near homeomorphism on $S(K,x)$ to a homeomorphic copy $\tilde{S}^{(n-2)}$ of $S^{(n-2)}$ implying that they are homeomorphic to each other. On the other hand this map is also cellular, since the metric segments and points of $S(K,x)$ are cellular sets in $S(K,x)$. To prove this, let $s = p_x^{-1}(v)$ be a segment in $S(K,x)$ for some $v \in \tilde{S}^{(n-2)}$. If now $Q \in s$ is a point, consider a metric ball $B_\epsilon(Q) \subset \mathrm{bd}(K)$ with center Q and radius $\epsilon > 0$ for which $\int(B_\epsilon(Q)) \cap S(K,x)$ is homeomorphic to $\mathbb{R}^{(n-2)}$. Such an $\epsilon > 0$ surely exists. In fact, Q has a neighborhood N_Q in $S(K,x)$ homeomorphic to $\mathbb{R}^{(n-2)}$. If for every ϵ we can choose a point $P_\epsilon \in B_\epsilon(Q) \cap S(K,x)$ which does not belong to N_Q then we have a sequence of points (P_ϵ) having the same property and tending to Q. Since N_Q is open in $S(K,x)$, this is impossible. Thus there is an $\epsilon > 0$ for which $B_\epsilon(Q) \cap S(K,x) = B_\epsilon(Q) \cap N_Q$. It implies that $\mathrm{int}(B_\epsilon(Q)) \cap S(K,x)$ is an open subset of N_Q relative to the topology of $S(K,x)$. Of course, ϵ depends on Q, but s is a compact set, thus there is a finite number of points Q_i and positive real numbers ϵ_i, such that for the minimal value ϵ^* of ϵ_i's we have $\cup \mathrm{int}(B_{\epsilon^*}(Q_i)) \supset s$. Here $\cup \mathrm{int}(B_{\epsilon^*}(Q_i))$ is the interior of the closed cell $\cup(B_{\epsilon^*}(Q_i))$. Since $B_\epsilon(Q) \cap S(K,x) = B_\epsilon(Q) \cap N_Q$ also holds for every ϵ' which is less or equal to ϵ, we have an infinite sequence of sets of form $\cup(B_{\epsilon^*}(Q_i))$ with the property needed to prove the cellularity of s.
Observe now that if $S(K,x)$ is an $(n-1)$-manifold with boundary then its boundary has two connected components which are equal to $\mathrm{bd}(\mathrm{cl}(K^+))$ and $\mathrm{bd}(\mathrm{cl}(K^-))$, respectively.
First we can see that $\mathrm{bd}(\mathrm{cl}(K^+))$ is the set of the common boundary points of $\mathrm{cl}(K^+)$ and $S(K,x)$ yielding $\mathrm{bd}(\mathrm{cl}(K^+)) \subset \mathrm{bd}(S(K,x))$. (We have $\mathrm{bd}(\mathrm{cl}(K^-)) \subset \mathrm{bd}(S(K,x))$, too.)
Secondly we note that there is no point of $\mathrm{int}(\mathrm{cl}(K^+))$ belonging to $S(K,x)$. Indirectly assume that the point P is in $\mathrm{int}(\mathrm{cl}(K^+)) \cap S(K,x)$. Then

- either one can find a neighborhood U of P in $S(K,x)$ which is homeomorphic to the $(n-1)$-dimensional half-space and therefore P is a boundary point of $\mathrm{cl}(K^+)$ (in U there exists a point Q with a neighborhood $V \subset S(K,x)$ homeomorphic to $\mathbb{R}^{(n-1)}$ such that $Q \in V \subset U$. It means that Q is a point of the complement of $\mathrm{cl}(K^+)$),
- or there is a neighborhood U homeomorphic to the space $\mathbb{R}^{(n-1)}$ for which $P \in U \subset S(K,x)$. In this case P is in the interior of $S(K,x)$ contradicting the assumption that it is a point of $\mathrm{int}(\mathrm{cl}(K^+))$.

In this way $\mathrm{int}(\mathrm{cl}(K^+)) = K^+$ and then $\mathrm{bd}(\mathrm{cl}(K^+)) = \mathrm{bd}(K^+)$ is the common boundary of K^+ and $S(K,x)$. Applying Lemma 2.1.7 we obtain that $\mathrm{bd}(\mathrm{cl}(K^+))$ is a connected closed subset of the boundary of $S(K,x)$.

Using the fact that $\mathrm{bd}(\mathrm{cl}(K^-))$ is the image of $\mathrm{bd}(\mathrm{cl}(K^+))$ by a central projection, we have a similar result for $\mathrm{bd}(\mathrm{cl}(K^-))$, too. (It is the common boundary of K^- and $S(K,x)$.) We will prove that the boundary of $S(K,x)$ is the disjoint union of these two sets.
The relation $\mathrm{bd}(S(K,x)) \subset \mathrm{bd}(\mathrm{cl}(K^-)) \cup \mathrm{bd}(\mathrm{cl}(K^+))$ is obvious. Consider a point P from the intersection $\mathrm{bd}(\mathrm{cl}(K^-)) \cap \mathrm{bd}(\mathrm{cl}(K^+))$. Let U be a neighborhood of P in $S(K,x)$. (It is homeomorphic to a half-space of $\mathbb{R}^{(n-1)}$.) Let B be a metric $(n-1)$-ball around P with such a sufficiently small radius $\epsilon > 0$, that the sets $B \cap U$ and $B \setminus (B \cap U)$ serve as topological images of a closed and the complementary open half-spaces of $\mathbb{R}^{(n-1)}$, respectively. (Similarly as the proof of the cellularity property of a segment goes one can show that such an $\epsilon > 0$ and ball B exist.) Since B contains points from each of the sets K^+ and K^- we have a contradiction by the separating property of $S(K,x)$. (There is no point of $S(K,x)$ in the complementary domain $B \setminus (B \cap U)$.)
This implies that the boundary of $S(K,x)$ has two connected components which are the common boundaries of $S(K,x)$ and K^+, $S(K,x)$ and K^-, respectively. Of course, these sets are also $(n-2)$-manifolds connected with straight line segments through all of their points. So we have that $S(K,x) = \mathrm{bd}(\mathrm{cl}(K^+)) \times [0,1]$ holds. We still have to prove that in this case $\mathrm{bd}(\mathrm{cl}(K^+))$ is homeomorphic to $S^{(n-2)}$, too. Since p_x on $\mathrm{bd}(\mathrm{cl}(K^+))$ into $S^{(n-2)}$ is also a cell-like (and cellular) mapping, $\mathrm{bd}(\mathrm{cl}(K^+))$ is an $(n-2)$-dimensional manifold and this restricted map is one to one, the last statement of the Theorem follows from Theorem 2.1.9, too. □

THEOREM 2.1.11 ([3]). *Let denote by $S(K,x)$ the shadow boundary of K in the direction x.*

I *$S(K,x)$ is an $(n-2)$-dimensional manifold if all of the non-degenerated general parameter spheres $\gamma_\lambda(K,x)$ with $\lambda > \lambda_0$ are $(n-2)$-dimensional manifolds, conversely if $S(K,x)$ is an $(n-2)$-dimensional manifold then all of the general parameter spheres are ANRs.*

II *$S(K,x)$ is an $(n-1)$-dimensional manifold with boundary if and only if there is a λ for which the general parameter sphere $\gamma_\lambda(K,x)$ is an $(n-1)$-dimensional manifold with boundary.*

To prove this theorem we used a theorem of M.Brown on the projective limit of compact metric spaces and corresponding near homeomorphisms (see [33]). The concept of the near homeomorphism of topological manifolds can be adapted to the case of compact metric spaces, too. A map from X to Y between compact metric spaces is a near homeomorphism if it is in the closure of the set of all homeomorphisms from X onto Y, with respect to the sup-norm metric on the space $C(X,Y)$ of all maps from X to Y. Now the mentioned theorem is:

THEOREM 2.1.12 (M.Brown). *Let (X_n) be an inverse sequence of compact metric spaces with limit X_∞. If all bonding maps $X_k \longrightarrow X_n$ are near homeomorphisms, then so are the limit projections $X_k \longrightarrow X_\infty$.*

Before the proof let us give an example showing that we should distinguish the above two cases.

EXAMPLE 2.1.6. Consider the union of the six connecting rectangles $\pm\{(r,1,t)|-1 \le r,t \le 1\}$, $\pm\{(r,s,t)|r+s=2, 1 \le r \le 2, -1 \le t \le 1\}$, $\pm\{(r,s,t)|r-s=2, 1 \le r \le 2, -1 \le t \le 1\}$ and the segments $\pm\{(r,0,2)|-\frac{3}{2} \le r \le \frac{3}{2}\}$. The convex hull K of this set is a convex polyhedron. If now the vector x is the position vector directed into the point $(4,0,0)$ we have three important values for the parameters of the generalized parameter spheres. For $\lambda_0 = 1$ the degenerated sphere $\gamma_{\lambda_0}(K,x)$ is a segment. For $1 < \lambda \le \frac{5}{4}$ the general parameter spheres $\gamma_\lambda(K,x)$ are homeomorphic to S^1. In the range $\frac{5}{4} < \lambda \le \frac{3}{2}$ the general parameter sphere $\gamma_\lambda(K,x)$ is a simplicial complex containing one or two-dimensional simplices, respectively. (This space is an ANR but is not a topological manifold.) Finally, in the last parameter domain $\lambda > \frac{3}{2}$ the set $\gamma_\lambda(K,x)$ is homeomorphic to the cylinder $S^1 \times [0,1]$. Since $S(K,x)$ is the union of six quadrangles, parallel to the x-axis it is also a cylinder. □

We think that true the following conjecture:

CONJECTURE 2.1.2. *If $S(K,x)$ is an $(n-2)$-dimensional manifold than all of the non-degenerated parameter spheres are also $(n-2)$-dimensional manifolds.*

Unfortunately we could not prove it.

PROOF. First we note that – for every $\lambda_0 < \lambda' < \infty$ – $S(K,x)$ can be considered as the inverse limit space X_∞ of the metric spaces $X_\lambda := \gamma_\lambda(K,x)$ for $\lambda' < \lambda$. In fact, by Lemma 2.1.4 if for $\lambda > \lambda_0$ the intersection of $\gamma_\lambda(K,x)$ by a longitudinal parameter curve, say r is a segment then $r \cap \gamma_\mu(K,x)$ with $\mu > \lambda$ is also a segment containing the segment $r \cap \gamma_\lambda(K,x)$. So in this case the union of the sets $r \cap \gamma_\mu(K,x)$ is the segment $r \cap S(K,x)$. On the other hand we have two possibilities for $r \cap \gamma_\lambda(K,x)$ being a point. First $r \cap S(K,x)$ is a point, too, meaning that for all $\mu > \lambda$ $r \cap \gamma_\lambda(K,x)$ is also a point. If now $r \cap S(K,x)$ is a segment then we have a value $\lambda' > \lambda$ with the property that if $\mu > \lambda'$ then $r \cap \gamma_\mu(K,x)$ is a segment, too. In this latter case $r \cap S(K,x) = \cup_{\mu \geq \lambda'} \{r \cap \gamma_\mu(K,x)\}$. Define now the left end of a segment parallel to x as the end having the smaller parameter in the usual parametrization with respect to x (meaning that a general point of a line parallel to x is written in the form $P + \tau x$ where P is a point of this line). Let us define the bonding map $p_{\lambda,\mu}$ for $\gamma_\mu(K,x)$ to $\gamma_\lambda(K,x)$ $(\mu > \lambda)$ in the following way: For a point P of $\gamma_\mu(K,x)$

$$p_{\lambda,\mu}(P) = \begin{cases} r \cap \gamma_\lambda(K,x) & \text{if } r \cap \gamma_\lambda(K,x) \text{ is a point} \\ P & \text{if } r \cap \gamma_\lambda(K,x) \text{ is a segment and } P \in r \cap \gamma_\lambda(K,x) \\ \text{the left end of } r \cap \gamma_\lambda(K,x) & \text{if } P \in r \cap \gamma_\mu(K,x) \setminus r \cap \gamma_\lambda(K,x) \end{cases} .$$

The continuity of this function (with respect to the relative metric) is obvious and the inverse (projective) limit space X_∞ can be identified with $S(K,x)$ by the limit mappings p_μ (defined in an analogous way from $S(K,x)$ to $\gamma_\mu(K,x)$ as the above functions $p_{\lambda,\mu}(P)$). (Of course, we have the sufficient equality $p_{\mu',\mu''} \circ p_{\mu'} = p_{\mu''}$ for $\mu'' > \mu'$.)
Using Theorems 2.1.9 and 2.1.12 above, the proof of the first direction of the first statement is an easy consequence. In fact, if for $\lambda > \lambda_0$ the space $\gamma_\lambda(K,x)$ is an $(n-2)$-manifold then using Theorem 2.1.9 we know that the bonding maps $p_{\mu',\mu''} : \gamma_{\mu''}(K,x) \longrightarrow \gamma_{\mu'}(K,x)$ are near homeomorphisms. By Theorem 2.1.12 we obtain that the limit projections p_λ are also near homeomorphisms. This implies that the space $S(K,x)$ is also an $(n-2)$ manifold.
Conversely, if now $S(K,x)$ is an $(n-2)$-dimensional manifold then it is locally contractible. By Lemma 2.1.4 this also implies that all of the general parameter spheres are locally contractible manifolds, too. On the other hand the general parameter spheres can be considered as the compact subsets of $\mathbb{R}^{(n-1)}$ meaning that they are ANRs. (See Theorem 8 p.117 in [**43**].)
The proof of both parts of the second statement uses Theorem 2.1.10. If first we have a general parameter sphere $\gamma_\lambda(K,x)$ which is an $(n-1)$-dimensional manifold with boundary then by Theorem 2.1.10 it is a cylinder with boundaries homeomorphic to $S^{(n-2)}$. In this case the shadow boundary contains this general parameter sphere showing that all point-inverses with respect to p_x are segments (with non-zero lengthes). On the other hand, the sets $\operatorname{bd} K^+ \cap S(K,x)$ and $\operatorname{bd} K^+ \cap \gamma_\lambda(K,x)$ coincide, showing that $S(K,x)$ is a cylinder based on an $(n-2)$ manifold homeomorphic to $S^{(n-2)}$. Since $\operatorname{bd} K^- \cap S(K,x)$ is homeomorphic to $S^{(n-2)}$ (by central symmetry) and these two sets are disjoint we close to that $S(K,x)$ is homeomorphic to $S^{(n-2)} \times [0,1]$, as we stated.
Conversely, if $S(K,x)$ is an $(n-1)$-manifold with boundary, then it is (by Theorem 2.1.10) homeomorphic to $S^{(n-2)} \times [0,1]$. Since this cylinder is compact there is a positive value ε less than or equal to the length of any segment intersected from the shadow boundary by a longitudinal parameter curve. This fact implies that there does exist a $\lambda < \infty$ such that $\gamma_\lambda(K,x) \subset S(K,x)$. The intersection $\gamma_\lambda(K,x) \cap K^+$ is the same as the intersection $S(K,x) \cap K^+$ which is one of the two components of the boundary of $S(K,x)$ homeomorphic to $S^{(n-2)}$. For this λ it is possible to find a trivial point-inverse with respect to the map p_x as we saw it in the example of this section, but for every $\lambda' > \lambda$ the general parameter sphere $\gamma_{\lambda'}(K,x)$ is a cylinder. Using now the fact that it is also the shadow boundary of a centrally symmetric convex body whose positive

part is the set K^+, we have proved that it is also a manifold with boundary homeomorphic to $S^{(n-2)} \times [0,1]$. □

A consequence of this result (if the bisector is a homeomorphic copy of $\mathbb{R}^{(n-1)}$ then the shadow boundary is a topological $(n-2)$-sphere) yields the proof of the first direction of the Conjecture 2.1.1. We have two more questions left concerning the same conjecture: Is the converse statement true or not? Is it possible that in the manifold case the embedding of the bisector and the shadow boundary are not standard ones? We prove here that the embedding of the examined sets (in the manifold case) are always standard ones, but the first question remains still open. The last step in the proof of the first direction of Conjecture 2.1.1 is the following theorem:

THEOREM 2.1.13 ([3]). *H_x is an $(n-1)$-dimensional manifold if and only if the non-degenerated general parameter spheres $\gamma_\lambda(K,x)$ are manifolds of dimension $(n-2)$.*

Since the neighborhoods of the point $\frac{1}{2}x$ (with respect to H_x) can not be homeomorphic to either R^n or a half space, this is the only manifold case for H_x.

PROOF. First we prove that if the non-degenerated general parameter spheres $\gamma_\lambda(K,x)$ are manifolds of dimension $(n-2)$ then H_x is an $(n-1)$-dimensional manifold. From Theorem 2.1.10 we know that the general parameter spheres are homeomorphic copies of $S^{(n-2)}$. Let us construct now the bisector H_x as the disjoint union of the sets $\lambda\gamma_\lambda(K,x)$ for $\lambda \geq \lambda_0$. The set $H_{x,\mu} = \{\lambda\gamma_\lambda(K,x) | \mu \geq \lambda \geq \lambda_0\}$ is obviously homeomorphic to $\gamma_\lambda(K,x) \cup K^+$ meaning that it is a homeomorphic copy of the closed $(n-1)$-dimensional ball. Thus int $H_{x,\mu}$ is homeomorphic to $\mathbb{R}^{n-1}$ for each $\mu \geq \lambda_0$. Applying now a theorem of M.Brown on chain of cells (see in [141] or [32]) saying that if a topological space is the union of an increasing sequence of open subsets, are homeomorphic to $\mathbb{R}^{(n-1)}$, resp. then it is also homeomorphic to $\mathbb{R}^{(n-1)}$, we get the required result.
Conversely, if H_x is homeomorphic to $\mathbb{R}^{(n-1)}$ then the projection $p_x : H_x \longrightarrow \mathbb{R}^{(n-1)}$ is a cellular map between two manifolds of the same dimension. Thus it is a near homeomorphism yielding that its restriction to the compact metric space $\lambda\gamma_\lambda(K,x)$ is a near homeomorphism, too. But its image is the boundary of a convex compact $(n-1)$-dimensional body so we get at once that it is a homeomorphic copy of $S^{(n-2)}$. Hence the general parameter spheres $\gamma_\lambda(K,x)$ for $\lambda > \lambda_0$ are manifolds of dimension $(n-2)$, as we stated. □

COROLLARY 2.1.2. *The proof of the first direction of the conjecture follows from the previous three theorems. In fact, if H_x is a topological hyperplane then each of the non-degenerated general parameter spheres is a homeomorphic copy of $S^{(n-2)}$ by Theorem 2.1.10 and Theorem 2.1.13. So by Theorem 2.1.11 we get that the shadow boundary is also a homeomorphic copy of $S^{(n-2)}$ which is the statement of the mentioned direction of our conjecture.*

On the other hand we could only prove in Theorem 2.1.11 that if $S(K,x)$ is a homeomorphic copy of $S^{(n-2)}$ then the non-degenerated general parameter spheres are ANRs, thus the manifold property for the bisector does not follow immediately from our theorems. Furthermore, in the manifold case we prove only that the bisector is a homeomorphic copy of $\mathbb{R}^{(n-1)}$ which is a weaker property as the required one. Consequently we have to investigate the question of embedding. In fact, all of the examples in geometric topology aiming a non-standard (wild) embedding of a set into $\mathbb{R}^n$ are based on the observation that the connectivity properties of the complement (with respect to $\mathbb{R}^n$) of the set can change if we apply a homeomorphism to it. In our case, for example, the complement of the bisector (which is now a homeomorphic copy of $\mathbb{R}^{(n-1)}$) is the disjoint union of homeomorphic copies of $\mathbb{R}^n$. It gives the chance to the existence of a homeomorphism on $\mathbb{R}^n$ to itself sending the bisector to a hyperplane. It is a well-known fact that a manifold homeomorphic to $S^{(n-1)}$ in S^n is unknotted if and only if the closures of the components its complement are homeomorphic copies of the closed n-cell $\mathcal{B}^n$. This implies that in the manifold case the embedding of the shadow boundary and the general parameter spheres are always standard. From this it follows the existence of a homeomorphism

of the boundary of K into itself sending these sets into a standard $(n-1)$-dimensional sphere of $\mathrm{bd}\, K$. Considering bisectors we have to carry out the proof in a bit more sophisticated way. Let φ be a homeomorphism sending H_x into $\mathbb{R}^{(n-1)}$ (which is now a hyperplane H of R^n). We consider the compactification of the embedding space by an element denoted by ∞. Extend first the map φ to the compact space $H_x \cup \{\infty\}$ by the condition $\varphi(\infty) = \infty$. Of course, this extended map gives a homeomorphism between the sets $H_x \cup \{\infty\}$ and $H \cup \{\infty\}$. Since the closure of the components of the complement of $H_x \cup \{\infty\}$ in $\mathbb{R}^n \cup \{\infty\}$ are closed n-cells the homeomorphism φ can be extended to a homeomorphism $\Phi : \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$. Since by our method we have: $\Phi(\infty) = \varphi(\infty) = \infty$ and $\Phi(H_x) = H$ we get that the bisector is a topological hyperplane as we stated. Thus the following statement has been proved:

THEOREM 2.1.14 ([3]). *In the manifold case the embedding of H_x, $S(K,x)$ and $\gamma_\lambda(K,x)$ are standard, respectively. This means that if the bisector is homeomorphic to $\mathbb{R}^{(n-1)}$ then it is a topological hyperplane.*

2.1.5. On bounded representation of bisectors (common work with H.Martini). Independently, H. Martini and S. Wu [118] introduced and investigated the concept of radial projection of bisectors. Strongly using the central symmetry of Minkowskian balls, they proved some interesting results on radial projections of bisectors.
Theorem 2.6 in [118] says that the shadow boundary is a subset of the closure of such a radial projection, and Theorem 2.9 there refers to the converse statement. If for a point x from the boundary of the unit ball there exists a point z, unique except for the sign, such that x is orthogonal to z in the sense of Birkhoff, then z is a point of the radial projection of the bisector corresponding to x and $-x$.
In a common paper with H. Martini [4] we introduced the concept of bounded representation of bisectors, which yields a useful combination of the notions of bisector, shadow boundary, and radial projection. We proved that the topological properties of the radial projection (in higher dimensions) do not determine the topological properties of the bisector. More precisely, the manifold property of the bisector does not imply the manifold property of the radial projection. The situation is different with respect to the bounded representation of the bisector. Namely, if one of them is a manifold, then the other is also. More precisely, if the bisector is a manifold of dimension $(n-1)$, then its bounded representation is homeomorphic to a closed $(n-1)$-dimensional ball B^{n-1} (i.e., it is a cell of dimension $(n-1)$). And conversely, if the bounded representation is a cell, then the closed bisector is also.
We will also presented new approaches to higher dimensional analogues of several theorems given in [118]. By our new terminology, we rewrote and reproved Theorems 2.6, 2.9, and 2.10 from that paper.
It is well known that there are different types of orthogonality in Minkowski spaces. In particular, for $x, y \in M^n$ we say that x is *Birkhoff orthogonal* to y if $\|x + ty\| \geq \|x\|$ for all $t \in \mathbb{R}$, denoted by $x \perp_B y$ (see [26]); and x is *isosceles orthogonal* to y if $\|x + y\| = \|x - y\|$, denoted by $x \perp_I y$ (cf. [92]). The *shadow boundary* $S(K,x)$ of K with respect to the direction x is the intersection of S and all supporting lines of K having direction x. Evidently, $S(K,x) = \{y \in \mathcal{S} : y \perp_B x\}$.
Given a point $x \in \mathcal{S}$, the *bisector* of $-x$ and x, denoted by $B(-x,x)$, consists of all those vectors y which are isosceles orthogonal to x with respect to the Minkowski norm generated by K. The *radial projection* $P(x)$ of this bisector consists of those points y of $\mathcal{S}$ for which there is a positive real value t such that $ty \in B(-x,x)$. In this subsection we denote by y the points of the unit sphere $\mathcal{S}$.
We remark that, in the relative topology of S, $P(x)$ can either be closed or open; this can be easily seen in the cases of the Euclidean and of the maximum norm. Thus, for topological investigations in higher dimensions we suggest the extension of the definition of $B(-x,x)$ to ideal points by a limit property.

DEFINITION 2.1.5 ([4]). *Consider the compactification of $\mathbb{R}^n$ to a closed ball B^n by the set of the common ideal points x_∞ ($-x_\infty \neq x_\infty$) of the parallel half-lines. We say that the point $y_\infty := \infty \cdot y$ is in the extracted bisector $B(-x,x)$ if there is a non-constant sequence $(t_i y_i) \in B(-x,x)$ for which $\lim_{i\to\infty} y_i = y$. We call the points of the original bisector* ordinary points *and the points added in this way* ideal points, *respectively.*

With this extended definition of $B(-x,x)$, $P(x)$ is closed. Let $P(x)^l$ be the collection of those points y of S for which $\|ty+x\| < \|ty-x\|$ holds, for all real $t \geq 0$. Let $P(x)^r$ denote the image of $P(x)^l$ under reflection at the origin.

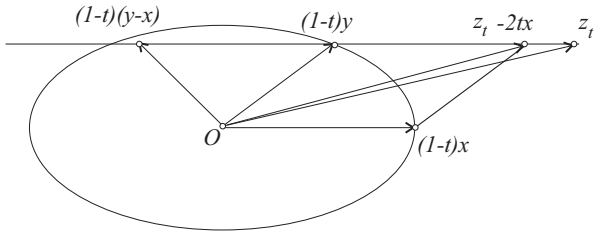

FIGURE 2.10. Vectors used in the proof of Proposition 2.1.1

PROPOSITION 2.1.1 ([4]). *In the described way, S is decomposed into three disjoint sets: $P(x)$, $P(x)^l$, and $P(x)^r$. $P(x)$ is an at least $(n-2)$-dimensional closed (and therefore compact) set in S which is connected for $n \geq 3$, the sets $P(x)^l$ and $P(x)^r$ are arc-wise connected components of their union.*

PROOF. By Theorem 5.1 of [118], $P(x)$ is connected for $n \geq 3$. We prove that it is also closed with respect to the relative topology of the boundary of the unit ball. To see this, consider a convergent sequence (y_i) in $P(x)$ having the limit y. For any i there is a new sequence of points (y_i^j) such that for every pair $\{i,j\}$ there are $t_j \in \mathbb{R}^+$ and $x_i^j \in B(-x,x)$ such that $(t_i^j y_i^j) = x_i^j$. (For an ordinary point the mentioned sequence can be regarded as a constant one.) It is clear that for the diagonal sequence (y_i^i) we have $\lim_{i\to\infty} y_i^i = y$, implying that y is also in $P(x)$. The continuity property of the norm function implies that all points of $\mathcal{S}$ belong to precisely one of the three mentioned sets. Thus the first statement is clear, and the union of $P(x)^l$ and $P(x)^r$ is open with respect to the topology of $\mathcal{S}$. Observe once more that $P(x)^l$ and $P(x)^r$ are images of each other regarding reflection at the origin. Furthermore, they are arc-wise connected sets. To prove this, consider the following inequality for an element y of $P(x)^r$:

$$\|(y - t(y-x) - x\| = (1-t)\|y-x\| < (1-t)\|y+x\| = \|(y - t(y-x)) + x - 2tx\|,$$

where $0 \leq t \leq 1$ is an arbitrary parameter. The point $z_t := (y - t(y-x)) + x = (1-t)y + (1+t)x$ is on the right half-line, starting with the point $(1-t)(y+x) = z_t - 2tx$ and being parallel to the vector x, meaning that its norm is larger than the norm of the point $z_t - 2tx$ (see Fig. 2.10). Thus $\|z_t\| \geq \|z_t - 2tx\|$, and so $\|(y - t(y-x)) - x\| < \|(y - t(y-x)) + x\|$.
A consequence of this inequality is that the arc of $\mathcal{S}$ connecting the respective endpoints of the vectors y and x belongs to the set $P(x)^r$. Thus every two points of $P(x)^r$ can be connected by an arc, as we stated. Now, with respect to the topology of their union, they are connected components. This means that both of them are also open with respect to the topology of $\mathcal{S}$. Thus $P(x)$ separates $\mathcal{S}$. By Aleksandrov's theorem (Theorem 5.12 in vol. I of [16]) we get that the topological dimension of $P(x)$ is at least $(n-2)$. □

The definition of the bounded representation of the bisector is:

DEFINITION 2.1.6 ([4]). *Let z be a point of $B(-x,x)$. If it is an ordinary point, then there is a unique value $1 \leq t_z < \infty$ for which $z \in (t_z\mathcal{S}+x) \cap (t_z\mathcal{S}-x)$. Let $\Phi : B(-x,x) \longrightarrow K$ denote the mapping which sends z into $\Phi(z) = \frac{1}{t_z} z$. We extend Φ to the ideal points by the following rule: The image of an ideal point is its radial projection. Denote the image set of Φ (with respect to*

this extended mapping) by $\Phi(B(-x,x))$. *We will call this set* the bounded representation of the bisector.

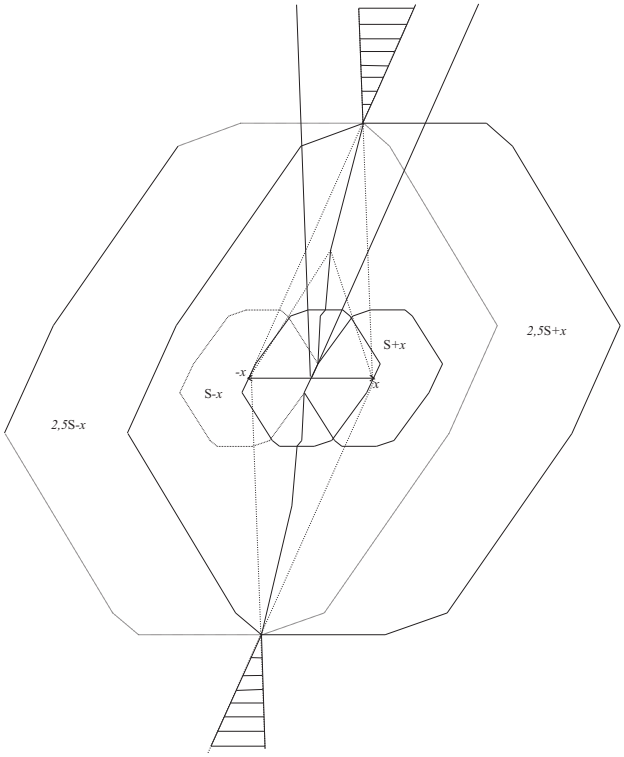

FIGURE 2.11. Bounded representation of the bisector

Geometrically the bounded representation of the bisector is well-handing as we can see from the following proposition:

PROPOSITION 2.1.2 ([4]). *The bounded representation of the bisector is the union of the shadow boundary of* K *and the locus of the midpoints of the chords of* K *parallel to* x.

PROOF. For an ordinary point z of the bisector we have $1 \leq t_z < \infty$, and thus the norm of $\frac{1}{t_z}z = \frac{1}{2}\left(\frac{1}{t_z}(z-x) + \frac{1}{t_z}(z+x)\right)$ is less or equal to 1. If it is equal to 1, then the point $\frac{1}{t_z}z$ is a point of a horizontal segment (parallel to x) of the boundary and thus a point of the shadow boundary, and the set of all points corresponding to the value t_z yields a horizontal segment of $\mathcal{S}$. If now $t \geq t_z$, the points of the bounded representation corresponding to this value t form another segment containing the segment of t_z. Thus the directions determined by the points of the segment of t_z are ideal points of the bisector, proving that the points of the shadow boundary are images of certain ideal points.
In the other case the obtained point is the midpoint of that chord whose endpoints are $\frac{1}{t_z}(z-x) \in \mathcal{S}$ and $\frac{1}{t_z}(z+x) \in \mathcal{S}$, respectively.
Now, by the definition of ideal points, the continuity of the mapping is clear. In fact, we have to check that the image of a point of the bisector with large norm is close to the boundary S of K. Since, by definition, t_z is equal to $\|z-x\|$, we have the two inequalities

$$1 \geq \|\frac{1}{t_z}z\| = \frac{\|z\|}{\|z-x\|} = \frac{1}{\|\frac{z}{\|z\|} - \frac{x}{\|z\|}\|} \geq \frac{1}{1+\frac{\|x\|}{\|z\|}},$$

showing that for z with large norm its bounded representation is close to $\mathcal{S}$. To visualize the proof, we show in Fig. 2.11 the bisector and its bounded representation in a two-dimensional space. □

James in [93] proved that a Minkowski space is Euclidean if and only if all of the bisectors contained in an $(n-1)$-dimensional subspace. Proposition 2.1.2 implies immediately the following

COROLLARY 2.1.3 ([4]). *The bounded representation of the bisector* $B(x,-x)$ *with respect to any point* x *from the unit sphere of a Minkowski space is contained in an* $(n-1)$*-subspace if and only if the Minkowski space is Euclidean.*

Finally we prove the following theorem:

THEOREM 2.1.15 ([4]). *If the bisector is a manifold of dimension* $(n-1)$ *with boundary, then its bounded representation is homeomorphic to the* $(n-1)$*-dimensional closed ball* B^{n-1}. *Conversely, if the bounded representation is a topological ball of dimension* $(n-1)$, *then the extracted bisector is of the same type. Furthermore, its relative interior (which is the set of its ordinary points) is a topological hyperplane of dimension* $(n-1)$.

PROOF. Assume that the bisector is a manifold of dimension $(n-1)$ with boundary. Then an ordinary point has a relatively open $(n-1)$-dimensional neighborhood in the bisector, and thus there are interior points. On the other hand, there is no ideal point which could be in the relative interior of the bisector implying that the set of ordinary points of the bisector is a manifold of dimension $(n-1)$. Hence our assumption implies that the shadow boundary $S(K,x)$ is a manifold of dimension $(n-2)$. In fact, from Theorem 2.1.13 and Theorem 2.1.11 we get that the shadow boundary is also a topological manifold of dimension $n-2$. Theorem 2.1.10 says that it is homeomorphic to S^{n-2}. On the other hand, the set C of midpoints of correspondingly directed chords containing interior points of K is always homeomorphic to the positive part S^+ of the boundary S of K, determined by the shadow boundary. Thus it is homeomorphic to $\mathbb{R}^{n-1}$. Finally we observe that the boundary of the latter set C is the shadow boundary itself, showing that the bounded representation of the bisector is homeomorphic to $\mathcal{B}^{n-1}$, as we stated.

We remark that the converse statement is true if and only if the manifold property of the bounded representation can be extended to the bisector. This is clear for the points mapping to the interior of K, but it is not evident for other points of the bisector. The problem is that the pre-images of a point of the shadow boundary could form a point or a half-line, respectively. Thus Φ is not an injective (but, of course, a surjective) continuous mapping. Clearly, both of the two sets (the bisector and its bounded representation) are continua, i.e., compact, connected Hausdorff (T_2) spaces. Moreover, the points and half-lines are cell-like sets; thus Φ is a cell-like mapping. Restricting Φ to the ideal point of the bisector, we get a bijective mapping onto the shadow boundary. We prove that the set of ideal points is compact in the bisector. It is a proper part I of $\mathcal{S}^{n-1}$ bounding the topological ball $\mathcal{B}^n$. Hence this point set can be regarded as a subset of an $(n-1)$-dimensional Euclidean space $\mathbb{R}^{n-1}$. (We can consider x_∞ as the center of a stereographic projection.) Its clear that I is bounded. It is also closed by its definition, and so it is compact by the Heine-Borel theorem on compact sets in $\mathbb{R}^{n-1}$. On the other hand, the shadow boundary can also be regarded as an $(n-2)$-sphere embedded into a Euclidean $(n-1)$-space, because x is not a point of it. A continuous and bijective mapping from a compact set of $\mathbb{R}^{n-1}$ into $\mathbb{R}^{n-1}$ is a homeomorphism (see again [**96**]). Thus the ideal points of the bisector give a topological $(n-2)$-dimensional sphere.

Now we prove that the ordinary points of the bisector are, with respect to its relative topology, interior points of it. We remark that it is trivial for a point $z \in B(-x,x)$ if $\Phi(z)$ is an interior point of K, because Φ (by its definition) is a homeomorphism on the collection of such points onto the interior of the bounded representation of the bisector. Thus it is also relatively open with respect to the bisector, and this part of the bisector is a topological manifold, homeomorphic to $\mathbb{R}^{n-1}$.

Let now $\Phi(z)$ belong to the shadow boundary. Since it is a topological sphere of dimension $n-2$, there is a cell of dimension $n-2$ (a homeomorphic copy of a closed ball of dimension $n-2$), namely Z, containing $\Phi(z)$ in its interior. The pre-image $\Phi^{-1}(\operatorname{int} B)$ of the interior $\operatorname{int} B$ of B is (by the continuity of Φ) open with respect to the topology of the bisector and contains z. Thus it has also an interior point with respect to the topology of the bisector.

Finally we observe that from the compactness of $\mathcal{B}$ the existence of an ε follows for which the set $\{v : \|z\| - \varepsilon \le \|v\| \le \|z\| - \varepsilon,\ v \in \Phi^{-1}(\mathcal{B})\}$ is a closed cone (truncated by two parallel surfaces) containing z in its interior. Since the interior of this body is homeomorphic to $\mathbb{R}^{n-1}$, we get that the set of ordinary points is a manifold of dimension $(n-1)$. In the proof of Theorem 2.1.13 it is shown that if the ordinary points of the bisector yield an $(n-1)$-manifold, then it is homeomorphic to $\mathbb{R}^{n-1}$, and Theorem 2.1.14 there establishes that it is a topological hyperplane. Thus we proved that the closed bisector is a cell of dimension $(n-1)$ whose interior can be embedded in the n-dimensional Euclidean space in a standard (unknotted) way, as we stated. □

2.2. Adjoint abelian operators and isometries

A generalization of the inner product and the inner product spaces (briefly i.p spaces) was raised by G. Lumer in [108].

DEFINITION 2.2.1 ([108]). *The semi inner-product (s.i.p) on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:*

s1: $: [x + y, z] = [x, z] + [y, z]$,
s2: $: [\lambda x, y] = \lambda[x, y]$ *for every* $\lambda \in \mathbb{C}$,
s3: $: [x, x] > 0$ *when* $x \neq 0$,
s4: $: |[x, y]|^2 \leq [x, x][y, y]$,
A vector space V with a s.i.p. is an s.i.p. space.

G. Lumer proved that an s.i.p space is a normed vector space with norm $\|x\| = \sqrt{[x, x]}$ and, on the other hand, that every normed vector space can be represented as an s.i.p. space. In [64] J. R. Giles showed that the following homogeneity property holds:

s5: : $[x, \lambda y] = \bar{\lambda}[x, y]$ for all complex λ.

This can be imposed, and all normed vector spaces can be represented as s.i.p. spaces with this property. Giles also introduced the concept of *continuous s.i.p. space* as an s.i.p. space having the additional property

s6: : For any unit vectors $x, y \in S$, $\mathrm{Re}\{[y, x + \lambda y]\} \to \mathrm{Re}\{[y, x]\}$ for all real $\lambda \to 0$.

The space is uniformly continuous if the above limit is reached uniformly for all points x, y of the unit sphere S. A characterization of the continuous s.i.p. space is based on the differentiability property of the space.

DEFINITION 2.2.2 ([64]). *A normed space is Gâteaux differentiable if for all elements x, y of its unit sphere and real values λ, the limit*

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda}$$

exists. A normed vector space is uniformly Frèchet differentiable if this limit is reached uniformly for the pair x, y of points from the unit sphere.

Giles proved in [64] that an s.i.p. space is a continuous (uniformly continuous) s.i.p. space if and only if the norm is Gâteaux (uniformly Frèchet) differentiable. In the second part of this dissertation we need a stronger condition on differentiability of the s.i.p. space. Therefore we define the differentiable s.i.p. as follows:

DEFINITION 2.2.3 ([8]). *A differentiable s.i.p. space is an continuous s.i.p. space where the s.i.p. has the additional property:*
s6': *For every three vectors x,y,z and real λ*

$$[x, \cdot]'_z(y) := \lim_{\lambda \to 0} \frac{\mathrm{Re}\{[x, y + \lambda z]\} - \mathrm{Re}\{[x, y]\}}{\lambda}$$

does exist. We say that the s.i.p. space is continuously differentiable, if the above limit, as a function of y, is continuous.

First we note that the equality $\mathrm{Im}\{[x, y]\} = \mathrm{Re}\{[-ix, y]\}$ together with the above property guarantees the existence and continuity of the complex limit: $\lim_{\lambda \to 0} \frac{[x, y+\lambda z]-[x,y]}{\lambda}$. Analogously to the theorem of Giles (see Theorem 3 in [64]) we combine this definition with the differentiability properties of the norm function generated by the s.i.p..

THEOREM 2.2.1 ([8], [9]). *An s.i.p. space is a (continuously) differentiable s.i.p. space if and only if the norm is two times (continuously) Gâteaux differentiable. The connection between the derivatives is*

$$\|y\|(\|\cdot\|''_{x,z}(y)) = [x, \cdot]'_z(y) - \frac{\mathrm{Re}[x, y]\mathrm{Re}[z, y]}{\|y\|^2}.$$

We need the following useful lemma going back, with different notation, to McShane [**119**] or Lumer [**109**].

LEMMA 2.2.1 ([**109**]). *If E is any s.i.p. space with $x, y \in E$, then*

$$\|y\|(\|\cdot\|'_x(y))^- \leq \mathrm{Re}\{[x,y]\} \leq \|y\|(\|\cdot\|'_x(y))^+$$

holds, where $(\|\cdot\|'_x(y))^-$ and $(\|\cdot\|'_x(y))^+$ denotes the left hand and right hand derivatives with respect to the real variable λ. In particular, if the norm is differentiable, then

$$[x,y] = \|y\|\{(\|\cdot\|'_x(y)) + \|\cdot\|'_{-ix}(y)\}.$$

Now we prove Theorem 2.2.1.

PROOF. To determine the derivative of the s.i.p., assume that the norm is twice differentiable. Then, by Lemma 2.2.1 above, we have

$$\frac{\mathrm{Re}\{[x,y+\lambda z]\} - \mathrm{Re}\{[x,y]\}}{\lambda} = \frac{\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|(\|\cdot\|'_x(y))}{\lambda} =$$

$$= \frac{\|y\|\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|^2(\|\cdot\|'_x(y))}{\lambda\|y\|} \geq \frac{|[y+\lambda z, y]|(\|\cdot\|'_x(y+\lambda z)) - \|y\|^2(\|\cdot\|'_x(y))}{\lambda\|y\|},$$

where we have assumed that the sign of $\frac{\|\cdot\|'_x(y+\lambda z)}{\lambda}$ is positive. Since the derivative of the norm is continuous, this follows from the assumption that $\frac{\|\cdot\|'_x(y)}{\lambda}$ is positive. Considering the latter condition, we get

$$\frac{\mathrm{Re}\{[x,y+\lambda z]\} - \mathrm{Re}\{[x,y]\}}{\lambda} \geq \|y\|^2 \frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda\|y\|} + \frac{\mathrm{Re}[z,y]}{\|y\|}\|\cdot\|'_x(y+\lambda z).$$

On the other hand,

$$\frac{\|y+\lambda z\|(\|\cdot\|'_x(y+\lambda z)) - \|y\|(\|\cdot\|'_x(y))}{\lambda} \leq \frac{\|y+\lambda z\|^2(\|\cdot\|'_x(y+\lambda z)) - |[y,y+\lambda z]|(\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|} =$$

$$= \frac{\|y+\lambda z\|^2(\|\cdot\|'_x(y+\lambda z)) - (\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|} + \lambda\mathrm{Re}[z,y+\lambda z]\frac{(\|\cdot\|'_x(y))}{\lambda\|y+\lambda z\|}.$$

Analogously, if $\frac{\|\cdot\|'_x(y)}{\lambda}$ is negative, then both of the above inequalities are reversed, and we get that the limit $\lim\limits_{\lambda\mapsto 0}\frac{\mathrm{Re}\{[x,y+\lambda z]\}-\mathrm{Re}\{[x,y]\}}{\lambda}$ exists, and equals to

$$\|y\|(\|\cdot\|''_{x,z}(y)) + \frac{\mathrm{Re}[x,y]\mathrm{Re}[z,y]}{\|y\|^2}.$$

Here we note that also in the case $\frac{\|\cdot\|'_x(y)}{\lambda} = 0$ there exists a neighborhood in which the sign of the function $\frac{\|\cdot\|'_x(y+\lambda z)}{\lambda}$ is constant. Thus we, need not investigate this case by itself. Conversely, consider the fraction

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda}.$$

We assume now that the s.i.p. is differentiable, implying that it is continuous, too. The norm is differentiable by the theorem of Giles. Using again Lemma 2.2.1 and assuming that $\frac{\mathrm{Re}[x,y]}{\lambda} > 0$, we have

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z) - (\|\cdot\|'_x(y))}{\lambda} = \frac{\mathrm{Re}[x,y+\lambda z]\|y\| - \mathrm{Re}[x,y]\|y+\lambda z\|}{\lambda\|y+\lambda z\|} =$$

$$= \frac{\mathrm{Re}[x,y+\lambda z]\|y\|^2 - \mathrm{Re}[x,y]\|y+\lambda z\|\|y\|}{\lambda\|y\|\|y+\lambda z\|} \leq \frac{\mathrm{Re}[x,y+\lambda z]\|y\|^2 - \mathrm{Re}[x,y]|[y+\lambda z,y]|}{\lambda\|y\|\|y+\lambda z\|} =$$

$$= \frac{\mathrm{Re}\{[x,y+\lambda z]\} - \mathrm{Re}\{[x,y]\}}{\lambda}\frac{\|y\|}{\|y+\lambda z\|} - \frac{\mathrm{Re}[x,y]\mathrm{Re}[z,y]}{\|y\|\|y+\lambda z\|}.$$

On the other hand, using the continuity of the s.i.p. and our assumption $\frac{\mathrm{Re}[x,y]}{\lambda} > 0$ similarly as above, we also get an inequality:

$$\|y\|\frac{\|\cdot\|'_x(y+\lambda z)-(\|\cdot\|'_x(y))}{\lambda} \geq \frac{\mathrm{Re}\{[x,y+\lambda z]\}-\mathrm{Re}\{[x,y]\}}{\lambda} - \frac{\mathrm{Re}[x,y+\lambda z]\mathrm{Re}[z,y+\lambda z]}{\|y+\lambda z\|^2}.$$

If we reverse the assumption of signs, then the direction of the inequalities will also change. Again a limit argument shows that the first differential function is differentiable, and the connection between the two derivatives is

$$\|y\|(\|\cdot\|''_{x,z}(y)) = [x,\cdot]'_z(y) - \frac{\mathrm{Re}[x,y]\mathrm{Re}[z,y]}{\|y\|^2}.$$

□

2.2.1. Characterization of adjoint abelian operators in Minkowski geometry. Stampfli in [136] has defined a bounded linear operator A to be *adjoint abelian* if and only if there is a duality map φ such that $A^*\varphi = \varphi A$. So evidently, A is adjoint abelian if and only if $A = A^T$, thus the adjoint abelian operators are in some sense "self-adjoint" ones. Lángi in [101] introduced the concept of the *Lipschitz property* of a semi inner product and investigated the diagonalizable operators of a Minkowski geometry $\{V, \|\cdot\|\}$. He said that the semi inner product $[\cdot,\cdot]$ has the Lipschitz property if for every x from the unit ball there is a real number κ such that for every y and z from the unit ball holds $|[x,y]-[x,z]| \leq \kappa\|y-z\|$. We note that from the differentiability property for the semi inner product (defined first in [8]) follows the Lipschitz property of the product, too. Let A be a diagonalizable linear operator of V, and let $\lambda_1 > \lambda_2 > \ldots \lambda_k \geq 0$ be the absolute values of the eigenvalues of A. If λ_i is an eigenvalue of A, then E_i denotes the eigenspace of A belonging to λ_i, and if λ_i is not an eigenvalue, set $E_i = \{0\}$. E_i defined similarly with $-\lambda_i$ in place of λ_i. The main result in [101] is the following.

THEOREM 2.2.2 ([101]). *Let V be a smooth finite-dimensional real Banach space such that the induced semi inner product $[\cdot,\cdot]$ satisfies the Lipschitz condition, and let $A : V \longrightarrow V$ be a diagonalizable linear operator. Then A is adjoint abelian with respect to $[\cdot,\cdot]$ if, and only if, the following hold.*

(1) $[\cdot,\cdot]$ is the direct sum of its restrictions to $\overline{E}_i = \mathrm{lin}\{E_i \cup E_{-i}\}$, $i = 1,\ldots,k$;
(2) for every value of i, the subspaces E_i and E_{-i} are both transversal and normal (meaning that they are mutually orthogonal in the sense of Birkhoff orthogonality);
(3) for every value of i, the restriction of A to $\overline{E}_i$ is the product of λ_i and an isometry of $\overline{E}_i$.

Using an observation from [8] and Corollary 3 from [101], we get that – by the assumption of the theorem – if no section of the unit sphere with a plane is an ellipse with the origin as its centre, then every diagonalizable adjoint abelian operator of X is a scalar multiple of an isometry of V. This motivates the following definition:

DEFINITION 2.2.4 ([5]). *A Minkowski n-space is* totally non-Euclidean *if it has no 2-dimensional Euclidean subspace.*

Now the corollary above says:

COROLLARY 2.2.1. *In a totally non-Euclidean Minkowski n-space every diagonalizable adjoint abelian operator is a scalar multiple of an isometry.*

The following theorem describe the structure of a real adjoint abelian operator.

THEOREM 2.2.3 ([5]). *Let V be a smooth finite-dimensional real Banach space with the induced semi inner product $[\cdot,\cdot]$. If A is adjoint abelian with respect to $[\cdot,\cdot]$ then V can be decomposed*

to the direct sum of A-invariant subspaces of dimension at most two. Restricting A to a 2-dimensional component it is a generalized dilatation *defined by the matrix*

$$\left[A|_{\mathrm{lin}\{a_s,b_s\}}\right]_{\{a_s,b_s\}} = |\lambda| \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \text{ where } |\lambda| \in \mathbb{R}^+ \text{ and } 0 < \varphi \leq 2\pi$$

and the basis $\{a_s, b_s\}$ holds the equalities $[a_s, a_s] = [b_s, b_s] = 1$, $[a_s, b_s] = [b_s, a_s] = 0$.

PROOF. First we prove that if A is an adjoint abelian operator and U is an A-invariant subspace then the orthogonal complement $U^\perp := \{v \in V \quad | \quad [v, u] = 0 \text{ for all } u \in U\}$ is also A invariant. In fact, for a $v \in U^\perp$ we have $[A(v), u] = [v, A(u)] = 0$ for all $u \in U$ proving this statement. From this it follows a decomposition of the space V to the direct sum of minimal invariant subspaces V_i with the property $V_i^\perp \supset V_j$ for all $j > i$. From the fundamental theorem of algebra it also follows that the dimension of V_i is at most 2.
Assume that Z is a 2-dimensional minimal invariant subspace of $A :_{\mathbb{R}} V \longrightarrow_{\mathbb{R}} V$ implying that it does not contain real eigenvector of A. Hence for every vector $z \in Z$ the pair of vectors z and $A(z)$ form a basis in Z. Thus the equality $A^2(z) = \gamma z + \delta A(z)$ also holds. Since this equation also valid if we substitute into $A(z)$ in the variable vector z we get that the polynomial equation $A^2 = \gamma I + \delta A$ holds on Z. Set $\delta = 2\alpha$ then we get the equation $(A - \alpha I)^2 = (\alpha^2 + \delta)I$. Since there is no real eigenvalue of A on Z we get that $(\alpha^2 + \delta) < 0$ say $-\beta^2$. Thus we have a polynomial equation of second order of form $(A - \alpha I)^2 = -\beta^2 I$ is valid on Z.
Let ${}_{\mathbb{C}}Z$ be the two dimensional complex vector space on the vectors of the additive commutative group Z, defined by the set of linear combinations

$$\{\xi f_1 + \zeta f_2 \quad \{f_1, f_2\} \text{ is a basis of } {}_{\mathbb{R}}Z \text{ and } \xi, \zeta \in \mathbb{C}\}$$

We can decompose the minimal polynomial $(x - \alpha)^2 + \beta^2$ to linear terms by the identity $(x - \alpha)^2 + \beta^2 = (x - \alpha - \beta i)(x - \alpha + \beta i)$. Hence we can correspond two complex eigenvalues $\lambda = \alpha + \beta i$ and $\overline{\lambda} = \alpha - \beta i$ of the extracted complex linear operator $\widetilde{A} :_{\mathbb{C}} Z \longrightarrow_{\mathbb{C}} Z$. (Note that with respect to the basis $\{f_1, f_2\}$ the complex operator $\widetilde{A}$ has the same (and real) coefficients as of the real linear operator A.) In ${}_{\mathbb{C}}Z$ for the eigenvalues λ and $\overline{\lambda}$ have distinct eigenspaces of dimension 1. These complex lines generated by the complex vectors

$$u = \xi f_1 + \zeta f_2 = (\alpha_1 + \beta_1 i)f_1 + (\alpha_2 + \beta_2 i)f_1 = (\alpha_1 f_1 + \alpha_2 f_2) + (\beta_1 f_1 + \beta_2 f_2)\, i =: a + bi,$$

and its conjugate $\overline{u} = a - bi$, respectively. (Here $a, b \in_{\mathbb{R}} Z$.) We say in this case that λ is a *complex eigenvalue of the real linear operator* A with *complex eigenvector* u. We identify the one-dimensional complex eigenspace of u with the two dimensional real subspace generated by a and b with the mapping $E :_{\mathbb{C}} < u > \longrightarrow_{\mathbb{R}} Z$

$$E((x + yi)(a + bi)) := \Re((x + yi)(a + bi)) + \Im((x + yi)(a + bi)) = (x + y)a + (x - y)b.$$

We note that E is a bijective mapping. In fact, if $x + y = x' + y'$ and $x - y = x' - y'$ then $x = x'$ and $y = y'$ and there is an unique solution of the equation system $r = x + y$ and $s = x - y$ it is $x = (r + s)/2$, $y = (r - s)/2$. From this follows that we can assume that a and b gives an Auerbach basis[2] of Z meaning in the rest part of this proof that $[a, a] = [b, b] = 1$ and $[a, b] = [b, a] = 0$.
Let now a complex eigenvalue of A is λ. Denote by E the complex eigenspace (of dimension d) corresponding to λ. Then $\overline{\lambda}$ is an eigenvalue with the eigenspace $\overline{E}$, where $\overline{E} = \{\overline{u} \quad u \in E\}$. If $\{u_1, \ldots, u_d\}$ is a complex basis of E then $\{\overline{u_1}, \ldots, \overline{u_d}\}$ is a basis of $\overline{E}$. Assuming that $u_s = a_s + b_s i$ and $\lambda = \alpha + \beta i$, we get that $\overline{u_s} = a_s - b_s i$ and $\overline{\lambda} = \alpha - \beta i$. Since

$$A(a_s) + A(b_s)i = A(u_s) = \lambda u_s = (\alpha a_s - \beta b_s) + (\beta a_s + \alpha b_s)\, i,$$

A is invariant on the real subspace $\widetilde{E} := \mathrm{lin}\{a_s, b_s \quad s = 1, 2, \ldots d\}$ which we call the real invariant subspace associated to λ. Its clear that for the eigenspace $\overline{E}$ we can associate the same invariant subspace. Since the vectors $u_s = a_s + b_s i \quad s = 1, \ldots, d$ form a basis of the

[2] See the precise definition before Theorem 3.1.2.

complex subspace E, the vectors $\{a_s, b_s \quad s = 1, \dots, d\}$ form a real generator system of $\widetilde{E}$ implying that the dimension is at most $2d$. Consider a pair of real vectors a_s, b_s. If $b_s = \lambda a_s$ then

$$a_s(\alpha - \lambda\beta) + a_s(\beta + \alpha\lambda)i = (a_s\alpha - b_s\beta) + (a_s\beta + \alpha b_s)\, i = A(a_s + b_s i) = (1 + i\lambda)A(a_s) =$$
$$= (1 + i\lambda)a_s(\alpha - \lambda\beta) = a_s(\alpha - \lambda\beta) + i\lambda a_s(\alpha - \lambda\beta),$$

implying that

$$\beta + \alpha\lambda = \lambda\alpha - \lambda^2\beta.$$

Since $\lambda \neq 0$ it follows that $\beta = 0$ which contradict by the fact that λ is not a real number. This shows that every pairs $\{a_s, b_s\}$ are independent vectors. Thus the complex eigenspace of dimension d is isomorphic to that real space of dimension $2d$ which is the direct product of its two dimensional subspaces generated by a_s and b_s.
Hence the adjoint abelian operator A invariant on the real plane $\mathrm{lin}\{a_s, b_s\}$ and with respect to the basis $\{a_s, b_s\}$ it has the matrix representation:

$$A = \begin{pmatrix} \alpha_r & \beta_r \\ -\beta_r & \alpha_r \end{pmatrix} = |\lambda| \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} =: |\lambda| F_\varphi.$$

where $|\cdot|$ means the absolute value of a complex number and φ is the argument of λ. □

We note that F_φ is also an adjoint abelian operator on that plane, we call it *generalized rotation* with respect to the basis $\{a_s, b_s\}$. In fact, $|\lambda| \neq 0$ because λ is not real. Thus we have

$$[F_\varphi(x), y] = \frac{1}{|\lambda|}\,[|\lambda|F_\varphi(x), y] = \frac{1}{|\lambda|}\,[x, |\lambda|F_\varphi(y)] = [x, F_\varphi(y)]\,.$$

EXAMPLE 2.2.1. To get a generalized rotation consider an inner product plane defined by the unit circle $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. The product is $[v, z] = [x_1 e + y_1 f, x_2 e + y_2 f] = \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2}$, and a required basis is $\{ae, bf\}$. The generalized rotation is in the Euclidean orthonormal basis $\{e, f\}$ is

$$F_\varphi = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \cos\varphi & \frac{b}{a}\sin\varphi \\ -\frac{a}{b}\sin\varphi & \cos\varphi \end{pmatrix}.$$

It is an isometry because it sends the unit disk into itself, however it is not adjoint abelian operator because of $[F_\varphi(e), f] = -\frac{a}{b^3}\sin\varphi \neq \frac{b}{a^3}\sin\varphi = [e, F_\varphi(f)]$. □

We suspect the following:

CONJECTURE 2.2.1 ([5]). *From Theorem 2.2.2 (or Theorem 1 (and Corollary 2) in [101]) we can omit the assumption "diagonalizable". More precisely every adjoint-abelian operator of a smooth Minkowski space is diagonalizable.*

In the case of l_p spaces this conjecture is true:

THEOREM 2.2.4 ([5]). *Let $1 < p < \infty$ be a real number. In a finite-dimensional real l_p space every adjoint abelian operator is diagonalizable.*

PROOF. Observe that for an l_2 space the statement is true because of the semi inner product is an inner product. Consider the Euclidean plane with the l_p norm $1 < p < \infty$. The corresponding semi inner product (see in [64]) can be defined by the equality

$$[z, v] = [x_1 a_s + y_1 b_s, x_2 a_s + y_2 b_s] = \frac{1}{\|s_2\|_p^{p-2}} \int_X s_1 |s_2|^{p-1} \mathrm{sgn}(s_2) d\mu =$$

$$= \frac{1}{\left(|x_2|^p + |y_2|^p\right)^{\frac{p-2}{p}}} \left(x_1 |x_2|^{p-1}\mathrm{sgn}(x_2) + y_1 |y_2|^{p-1}\mathrm{sgn}(y_2)\right),$$

where $\{a_s, b_s\}$ is an orthonormal basis in the Euclidean sense and Auerbach basis with respect to the l_p norm associated to the above product. Now we have the formulas

$$[F_\varphi(z), v] = \frac{1}{\left(|x_2|^p + |y_2|^p\right)^{\frac{p-2}{p}}} \left((\cos\varphi x_1 + \sin\varphi y_1)|x_2|^{p-1}\mathrm{sgn}(x_2) + (\cos\varphi y_1 - \sin\varphi x_1)|y_2|^{p-1}\mathrm{sgn}(y_2)\right),$$

and

$$[z,F_\varphi(v)]=\frac{\left(x_1|\cos\varphi x_2+\sin\varphi y_2|^{p-1}\mathrm{sgn}(\cos\varphi x_2+\sin\varphi y_2)+y_1|\cos\varphi y_2-\sin\varphi x_2|^{p-1}\mathrm{sgn}(\cos\varphi y_2-\sin\varphi x_2)\right)}{\left(|\cos\varphi x_2+\sin\varphi y_2|^p+|\cos\varphi y_2-\sin\varphi x_2|^p\right)^{\frac{p-2}{p}}}.$$

For $\varphi=\pi/2$ we get that

$$[F_\varphi(z),v]=\frac{1}{\left(|x_2|^p+|y_2|^p\right)^{\frac{p-2}{p}}}\left(y_1|x_2|^{p-1}\mathrm{sgn}(x_2)+(-x_1)|y_2|^{p-1}\mathrm{sgn}(y_2)\right)=-[z,F_\varphi(v)]$$

holds for all z and v. Since $[F_\varphi(z),v]=[z,F_\varphi(v)]$ also holds for all z and v, we get that $F_\varphi(z)=0$ for all z giving a contradiction. Thus $\varphi\neq\pi/2$ for an adjoint abelian generalized rotation.
If $\varphi=\pi$ then $F_\varphi(v)=-v$ and it is diagonalizable for all p.
Finally if $\varphi=3\pi/2$ then

$$[F_\varphi(z),v]=\frac{1}{\left(|x_2|^p+|y_2|^p\right)^{\frac{p-2}{p}}}\left(y_1|x_2|^{p-1}\mathrm{sgn}(x_2)+x_1|y_2|^{p-1}\mathrm{sgn}(y_2)\right)$$

and

$$[z,F_\varphi(v)]=\frac{1}{\left(|x_2|^p+|y_2|^p\right)^{\frac{p-2}{p}}}\left(y_1|x_2|^{p-1}\mathrm{sgn}(x_2)-x_1|y_2|^{p-1}\mathrm{sgn}(y_2)\right)$$

providing the strict inequality $[F_\varphi(z),v]>[z,F_\varphi(v)]$ for z and v with positive x_1 and y_2. This is a contradiction, too.
For general (and fixed) φ we get the equality

$$\left(|\cos\varphi x_2+\sin\varphi y_2|^p+|\cos\varphi y_2-\sin\varphi x_2|^p\right)^{\frac{p-2}{p}}\left((\cos\varphi x_1+\sin\varphi y_1)\,|x_2|^{p-1}\mathrm{sgn}(x_2)+\right.$$
$$\left.+(\cos\varphi y_1-\sin\varphi x_1)|y_2|^{p-1}\mathrm{sgn}(y_2)\right)=\left(|x_2|^p+|y_2|^p\right)^{\frac{p-2}{p}}\left(x_1|\cos\varphi x_2+\sin\varphi y_2|^{p-1}\mathrm{sgn}\left(\cos\varphi x_2+\sin\varphi y_2\right)+\right.$$
$$\left.+y_1|\cos\varphi y_2-\sin\varphi x_2|^{p-1}\mathrm{sgn}(\cos\varphi y_2-\sin\varphi x_2)\right),$$

which holds for all z and v.
First we substitute $x_2=y_2$ and $y_1=0$ into this equality and we get:

$$|x_2|^{2p-3}\left(|\cos\varphi+\sin\varphi|^p+|\cos\varphi-\sin\varphi|^p\right)^{\frac{p-2}{p}}x_1\mathrm{sgn}(x_2)(\cos\varphi-\sin\varphi)=$$
$$=|x_2|^{2p-3}|\cos\varphi+\sin\varphi|^{p-1}x_1\mathrm{sgn}(x_2)\mathrm{sgn}(\cos\varphi+\sin\varphi),$$

implying the other equality

$$\left(|\cos\varphi+\sin\varphi|^p+|\cos\varphi-\sin\varphi|^p\right)^{\frac{p-2}{p}}(\cos\varphi-\sin\varphi)=|\cos\varphi+\sin\varphi|^{p-1}\mathrm{sgn}(\cos\varphi+\sin\varphi).$$

From this immediately follows that either $\cos\varphi\pm\sin\varphi>0$ or $\cos\varphi\pm\sin\varphi<0$.
We can also substitute the equalities $y_2=0$ and $x_1=y_1$ into the original equality. This leads to the equality:

$$\left(|\cos\varphi|^p+|-\sin\varphi|^p\right)^{\frac{p-2}{p}}(\cos\varphi+\sin\varphi)=|\cos\varphi|^{p-1}\mathrm{sgn}(\cos\varphi)+|-\sin\varphi|^{p-1}\mathrm{sgn}(-\sin\varphi).$$

Now from the assumption $\cos\varphi\pm\sin\varphi>0$ it follows that $\mathrm{sgn}(\cos\varphi)=1$ and we have two possibilities. If $\mathrm{sgn}(-\sin\varphi)=-1$ then we get

$$\left(1+(\tan\varphi)^p\right)^{\frac{p-2}{p}}(1+\tan\varphi)=1-(\tan\varphi)^{p-1}.$$

Let $f(p):=\left(1+(\tan\varphi)^p\right)^{\frac{p-2}{p}}(1+\tan\varphi)-1+(\tan\varphi)^{p-1}$ be a function of p for a fixed admissible φ. It is clear that $\lim_{p\to\infty}f(p)=\tan\varphi$ and a short calculation shows that for $p>2$ it is a non-increasing function which at $p=2$ is $2\tan\varphi$ hence for $p\geq 2$ we get that $f(p)>0$. The function $f(p)$ on the interval $1<p<2$ is concave showing that $f(p)\geq\min\{f(1),f(2)\}>0$. Thus there is no p and φ for which this equality can be hold.
If $\mathrm{sgn}(-\sin\varphi)=1$ then we get the equality

$$\left(1+|\tan\varphi|^p\right)^{\frac{p-2}{p}}(1-|\tan\varphi|)=1+|\tan\varphi|^{p-1},$$

and the function

$$f(p):=1+|\tan\varphi|^{p-1}-\left(1+|\tan\varphi|^p\right)^{\frac{p-2}{p}}(1-|\tan\varphi|)>1+|\tan\varphi|^{p-1}-1-|\tan\varphi|^p$$

is a positive one for $1 < p < \infty$, since $|\tan\varphi| < 1$.
Thus remains only one possibility which could give a non-trivial adjoint abelian generalized rotation in an l_p space (for certain p) when we assume that $\cos\varphi \pm \sin\varphi < 0$. In this case $\operatorname{sgn}(\cos\varphi) = -1$ and $|\cos\varphi| > |\sin\varphi|$. However in this case the substitution $y_2 = 0$ and $x_1 = y_1$ leads to the same equalities as in the previous one leading to the same contradictions. Thus there is no non-diagonalizable adjoint abelian generalized rotation in an real l_p space of finite dimension, as we stated. □

We note that in the case of a Minkowski geometry we got a new proof for the known fact that every adjoint abelian operator on L_p $(1 < p < \infty, \quad p \neq 2)$ is a multiply of an isometry (see in [61]).

2.2.2. Characterization of isometries in Minkowski geometry. A Banach space isometry is a linear mapping which preserves the norm of the vectors. As it can be seen easily, the following theorem holds.

THEOREM 2.2.5 ([97]). *A mapping in a smooth Banach space is an isometry if and only if it preserves the (unique) s.i.p..*

Thus, if the norm is at least smooth, then the two types of linear isometry coincide. On the basis of the results of Stampfli [136] we have two corollaries:

COROLLARY 2.2.2 ([97]). *In any smooth uniformly convex Banach space, U is an invertible isometry if and only if $U^{-1} = U^T$. As a result if in addition $U^{-1} = U$ then U is scalar.*

Stampfli has defined an operator U to be *iso-abelian* if and only if there is a duality map ϕ such that $\phi U = (U^*)^{-1}\varphi$.

COROLLARY 2.2.3 ([97]). *In a smooth Banach space U is iso-abelian if and only if it is an invertible isometry.*

The above statement was extended to include the non-smooth case in [98]. Precisely:

THEOREM 2.2.6 ([98]). *Let V be a normed linear space (real or complex) and U be an operator mapping V into itself. Then U is an isometry if and only if there is a semi inner product $[\cdot,\cdot]$, such that $[U(x), U(y)] = [x, y]$ for all x and y.*

As a corollary of this theorem was proven the following:

COROLLARY 2.2.4 ([98]). *U is iso-abelian if and only if it is an invertible isometry.*

For our characterization important the following result:

THEOREM 2.2.7 ([98]). *A finite dimensional eigenspace of an isometry has a complement invariant under the isometry.*

For the construction can be seen that this complement is orthogonal to the given eigenspace of the isometry with respect to that semi inner product which preserved by the isometry. Since every linear mapping there is at least one (complex) eigenvalue hence a complex finite-dimensional Banach space is an orthogonal direct sum of eigenspaces of a given isometry (See Corollary 4 in [98].) For the real case we get analogously the following statement:

THEOREM 2.2.8 ([5]). *Let V be a finite dimensional real Banach space, $U : V \longrightarrow V$ be an isometry on V, and $[\cdot,\cdot]$ is a semi inner product preserved by the isometry U. Then there is a decomposition of the space of form $V = V_1 \oplus \ldots V_s \oplus V_{s+1} \oplus \ldots \oplus V_l \oplus V_{l+1} \oplus \ldots \oplus V_{l+k}$, where V_i $1 \leq i \leq l$ are U-invariant mutually orthogonal eigenspaces of dimension 1 if $1 \leq i \leq s$ the corresponding eigenvalue is 1 and for $s \leq i \leq l$ the common eigenvalue is -1; moreover $(n-l)$ is even and the subspaces $V_{l+1}, \ldots, V_{l+k}$ are 2-dimensional U-invariant subspaces all of them*

are orthogonal to the 1*-dimensional ones. Restricting* U *to a* 2*-dimensional component it is a* generalized rotation *with respect to an Auerbach basis* $\{a_s, b_s\}$ *defined by the matrix*

$$\left[A|_{\mathrm{lin}\{a_s,b_s\}}\right]_{\{a_s,b_s\}} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \text{ where } 0 < \varphi \leq 2\pi$$

PROOF. Since V is an orthogonal direct sum of the eigenspaces of U we have n mutually orthogonal eigenvectors of U, say $u_1, \ldots, u_n$. Since X is a finite dimensional real Banach space the eigenvalues $\lambda_1, \ldots, \lambda_l$ corresponding to $u_1, \ldots, u_l$ are real numbers and the rest eigenvalues $\lambda_{l+1}, \ldots, \lambda_n$ are complex ones.
First examine the eigenvalues $\lambda_1, \ldots, \lambda_l$. Since U is an isometry we have only two possibilities for its values, these are 1 and -1. We can assume that $\lambda_1 = \cdots = \lambda_s = 1$ and $\lambda_{s+1} = \cdots = \lambda_l = -1$. In the subspace generated by the first s eigenvectors every vectors are eigenvectors with eigenvalue 1 thus we can choose $u_1, \ldots, u_s$ as the elements of an Auerbach basis (hence there are mutually orthogonal vectors). We choose the basis $\{u_{s+1}, \ldots, u_l\}$ analogously from the eigenspace of eigenvalue -1. Since two eigenvectors corresponding to distinct eigenvalues are mutually orthogonal to each other, we get the orthogonality property of the statement about the first l eigenspaces.
Assume now that $\lambda_{l+(2r-1)} = \overline{\lambda}_{l+2r}$ holds for $r = 1, \ldots, (n-l)/2$. Consider again the vectors $u_{l+(2r-1)} = a_{l+(2r-1)} + b_{l+(2r-1)}i$ and scalars $\lambda_{l+(2r-1)} = \alpha_{l+(2r-1)} + \beta_{l+(2r-1)}i$ such that $U(u_{l+(2r-1)}) = \lambda_{l+(2r-1)}u_{l+(2r-1)}$. (See the analogous construction in the proof of Theorem 2.2.3 on adjoint abelian operators.) The real subspaces $\mathrm{lin}\{a_{l+(2r-1)}, b_{l+(2r-1)}\}$ are invariant with respect to U and have dimension 2. Since $\lambda_{l+2r} = \alpha_{l+(2r-1)} - \beta_{l+(2r-1)}i$ and $u_{l+2r} = a_{l+2r} + b_{l+2r}i = a_{l+(2r-1)} - b_{l+(2r-1)}i$ we also have that $\mathrm{lin}\{a_{l+2r}, b_{l+2r}\} = \mathrm{lin}\{a_{l+(2r-1)}, b_{l+(2r-1)}\}$. Hence $V_{l+(2r-1)} = V_{l+2r} = \mathrm{lin}\{a_{l+(2r-1)}, b_{l+(2r-1)}\}$ is an eigenspace of dimension at most 2. The case, when $b_{l+(2r-1)} = \alpha a_{l+(2r-1)}$ with real α implies that $a_{l+(2r-1)}$ is a real eigenvector with complex eigenvalue λ_i is impossible thus we get the decomposition of the statement. Since the equality $[a_{l+(2r-1)} + b_{l+(2r-1)}i, u_r] = 0$ implies the respective equalities $[a_{l+(2r-1)}, u_r] = 0$ and $[b_{l+(2r-1)}, u_r] = 0$, the last statement on orthogonality is also true. Finally from the U-invariant property it follows that U restricted to a 2-dimensional invariant subspace is a generalized dilatation (see Theorem 2.2.3). On the other hand U is an isometry thus $|\lambda_{l+(2r-1)}| = 1$ for all r hence it is a general rotation as we stated. □

REMARK 2.2.1. *We note that there are non-diagonalizable general rotations which are also isometries. In an* l_p *space of dimension* 2 *for the general rotation* $F_{\pi/2}$ *we get* $F_{\pi/2}(x_1a_s + y_1b_s) = (y_1a_s - x_1b_s)$ *and* $F_{\pi/2}(x_2a_s + y_2b_s) = (y_2a_s - x_2b_s)$ *showing that*

$$[F_{\pi/2}z, F_{\pi/2}v] = \frac{1}{(|y_2|^p + |-x_2|^p)^{\frac{p-2}{p}}}\left(y_1|y_2|^{p-1}\mathrm{sgn}(y_2) - x_1|-x_2|^{p-1}\mathrm{sgn}(-x_2)\right) =$$

$$= \frac{1}{(|x_2|^p + |y_2|^p)^{\frac{p-2}{p}}}\left(y_1|y_2|^{p-1}\mathrm{sgn}(y_2) + x_1|x_2|^{p-1}\mathrm{sgn}(x_2)\right) = [z, v].$$

2.2.3. The group of isometries. In geometric algebra, one studies the properties of certain algebraic entities that can be directly linked with geometric objects, and analyses how their (algebraic) properties relate to geometric properties of the underlying geometry under investigation. This approach will be applied here to the study of "strictly convex" Minkowski spaces. It is particulary interesting to characterize their group of isometries or related transformation groups. Although the lines of strictly convex non-Euclidean Minkowski planes are just their affine lines, the group of their isometries is small. Namely, it is the semi-direct product of the translation group by a finite group of even order which either consists of Euclidean rotations or is the dihedral group. This nice fact was proven by several authors (see in [62],[140] and [114]).

THEOREM 2.2.9 ([62],[140],[114]). *If* $(V, \|\cdot\|)$ *is a Minkowski plane that is non-Euclidean, then the group* $\mathcal{I}(2)$ *of isometries of* $(V, \|\cdot\|)$ *is isomorphic to the semi-direct product of the*

translation group $\mathcal{T}(2)$ of $\mathbb{R}^2$ with a finite group of even order that is either a cyclic group of rotations or a dihedral group.

In higher dimension it is possible for the group of linear isometries to be infinite without the space being Euclidean (e.g. if the unit ball is a elliptic cylinder in $\mathbb{R}^3$). The proof can be found in [**140**] uses the concept of Löwner-John's ellipsoids. John's (Löwner) ellipsoid of the unit ball C is the unique ellipsoid with maximal (minimal) volume contained (circumscribed) in (about) it. It is clear that every isometries which leave invariant the unit ball is also send these ellipsoids into themselves, respectively. A nice consequence of this fact (proved first by Auerbach in [**18**]) is the following:

COROLLARY 2.2.5 ([**140**],[**18**]). *If the isometry group of a Minkowski space is transitive on the unit ball of the space then the unit ball is ellipsoid and a space is Euclidean.*

On the other hand Gruber in [**73**] shows that for "most" cases the group of isometries is finite. It follows from the fact, that in "most" cases a Minkowski unit ball meets the boundary of the Löwner ellipsoid in $d(d+1)/2$ pairs of symmetric points. (See in [**73**].) Using again the concept of John's ellipsoid we can prove a similar result which is also a generalization of Theorem 2.2.9.

THEOREM 2.2.10 ([**5**]). *If the unit ball $\mathcal{B}$ of $(V, \|\cdot\|)$ has no intersection with a two-plane which is an ellipse, then the group $\mathcal{I}(3)$ of isometries of $(V, \|\cdot\|)$ is isomorphic to the semi-direct product of the translation group $\mathcal{T}(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

PROOF. Since at any point of V there exists a point reflection that is an isometry of $(V, \|\cdot\|)$, the group $\mathcal{I}(n)$ contains the semi-direct product of $\mathcal{T}(n)$ with a point reflection. Since $\mathcal{I}(n)$ is a closed subgroup of the Lie group of the affinities, the translation group $\mathcal{T}(n)$ is a normal subgroup of $\mathcal{I}(n)$ and $\mathcal{I}(n)$ is a semi-direct product of $\mathcal{T}(n)$ with the stabilizer $\mathcal{I}(n)_0$ of the point 0 in $\mathcal{I}(n)$ leaving the unit ball $\mathcal{B}$ invariant. On the other hand every isometry of V is also an affine isometry thus the elements of $\mathcal{I}(n)_0$ are in the special linear group of order n, too (see [**62**]).
For $n = 3$ from Theorem 2.2.8 we get that an isometry has at least one eigenvector and we have two possibilities, either it is diagonalizable operator or it is not. In the second case it has a minimal invariant subspace of dimension 2. Let $\mathcal{I}_x$ be the subgroup of $\mathcal{I}(3)_0$ containing those isometries which fixed the 1-dimensional subspace of x. Then the 2-dimensional subspace orthogonal to x is also invariant with respect to the elements of $\mathcal{I}_x$ (see Theorem 2.2.8). By Theorem 2.2.9 the group $\mathcal{I}_x$ is a finite of even order that is a cyclic group or a dihedral group. Consider now the John's ellipsoid E ([**140**]) of the unit ball $\mathcal{B}$. The concept of John's ellipsoid is affine invariant hence without loss of generality we can assume that E is an Euclidean ball inscribed into the suitable affine copy of $\mathcal{B}$ (which for simplicity we also denote by $\mathcal{B}$). (Now the investigated isometries are elements of $O(3)$.) Consider the group G of elements of $\mathcal{I}(3)_0$ belonging to $SO(3)$. Taking into consideration that the "determinant" map $\det : \mathcal{I}(3)_0 \to \{\pm 1\}$ is a surjective group homomorphism whose kernel G has index 2 in $\mathcal{I}(3)_0$, so that, G is finite if and only if $\mathcal{I}(3)_0$ is so. Let a point x is a common point of the boundary $\mathcal{S}$ of $\mathcal{B}$ and the boundary of E. (Of course such a point is exist.) Let denote by $\mathcal{S}^+$ the closed half sphere containing x and bounded by the hyperplane orthogonal to x through the origin. If the group G is infinite then the orbit of x is also contains infinitely many distinct points of form $T_i(x) \in \operatorname{bd} E \cap \mathcal{S}^+$ where $T_i \in \mathcal{I}(3)_0$. Since $\operatorname{bd} E \cap \mathcal{S}^+$ is compact for every $k \in \mathbb{N}$ there is two indices $i \neq j$ such that $\|T_i(x) - T_j(x)\| \leq 1/k$ implying that $\|T_j^{-1}T_i(x) - x\| \leq 1/k$. Consider the isometry $T_j^{-1}T_i \in SO(3)$. Hence $T_j^{-1}T_i$ is rotation about an axis say x_k. Thus the points $\left(T_j^{-1}T_i\right)^l(x)$ for $l \in \mathbb{N}$ are on a two dimensional intersection of $\operatorname{bd} E$, so they are also on a circle E_k. This circle through the point x contains a set of points of $\mathcal{S}$ with successive distance at most $1/k$ forming an $1/k$-net on it. Let denote by y_k the unit normal vector of the plane of E_k directed by $\mathcal{S}^+$. The set $Y := \{y_k \quad k \in \mathbb{N}\}$ is infinite and hence it has a convergent subsequence (y_{k_i}) with limit y. Consider now the circle $E(x, y)$ defined by the intersection of E with the plane

through x and orthogonal to y. It has the property that if $z \in E(x,y)$ that for every $\varepsilon > 0$ there is a point u of $\operatorname{bd} E \cap \mathcal{S}$ such that $\|z-u\| \leq \varepsilon$. This implies that $E(x,y) \subset \operatorname{bd} E \cap \mathcal{S}$ giving a contradiction with our assumption. Thus the group $\mathcal{I}(n)_0$ is finite and the statement is true. □

REMARK 2.2.2. *We note that we proved the finiteness of the point group from a stronger assumption that of the "totally non-Euclidean" property. A method using Löwner-John ellipsoids can not be applied to prove a more general statements on this direction because of there are Minkowski spaces which are totally non-Euclidean but the intersection of the John's ellipsoid of its unit sphere contains ellipse. For a simple example, consider an Euclidean unit ball $\mathcal{B}$ and one of its great circle S. Let $H(2n,\varepsilon)$ be a regular polygon circumscribed to $(1+\varepsilon)S$ with $2n$ vertices. Now define the unit ball $\mathcal{B}(n,\varepsilon) := \operatorname{conv}\{\mathcal{B} \cup H(2n,\varepsilon)\}$. It is clear that the Minkowski space with unit ball $\mathcal{B}(n,\varepsilon)$ is totally non-Euclidean however for small ε and for large n the John's ellipsoid of $\mathcal{B}(n,\varepsilon)$ is $\mathcal{B}$, hence $\operatorname{bd} C(n,\varepsilon) \cap \operatorname{bd} \mathcal{B}$ contains circle.*

This motivates the following problem:

PROBLEM 2.2.1. *Is it true or not that if for $n \geq 3$ the Minkowski n-space is totally non-Euclidean one (see Definition 2.2.4) then its isometry group $\mathcal{I}(n)$ is a semi-direct product of the translation group $\mathcal{T}(n)$ with a finite subgroup of $SL(n)$?*

2.3. Conics and roulettes in Minkowski planes

The following section contains investigations on two types of constructive curves in Minkowski plane. The two subsections contains the results of two papers the first one in common with H. Martini ([6]) and the second one is common with V. Balestro and H. Martini ([7]).

2.3.1. Conics (Common work with H. Martini). Now we turn out to conics in a Minkowski normed space. With H. Martini we presented in [6] a systematic investigation of possible definitions of conics extended to normed (or Minkowski) planes. In the Euclidean situation the metric definitions of conics and the analytic one, namely defining them as family of curves of second order, clearly yield the same type of curves; so we have various different definitions of the same class of curves. In normed planes neither the metric definitions nor the analytic one yield the same type of curves. Furthermore, it is not clear what the notions "curve of second order", "cone of second order" or "sections of a cone" mean. We considered the usual metric definitions of conics in the Euclidean plane, adopt them for normed planes and list various properties of the resulting classes of curves. In normed planes we have three different possibilities to define ellipses metrically. Before [6], only the first one was investigated (see [146]). So the following definitions refer to an "ellipse" in a normed plane X.

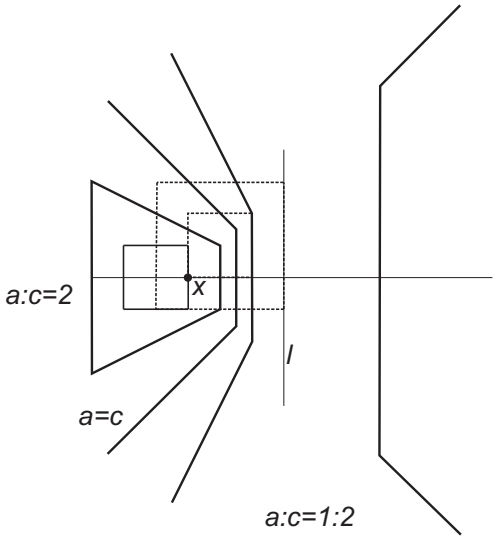

FIGURE 2.12. Conics on the l_∞ plane

DEFINITION 2.3.1 (based on foci, [6], [146]). *Let $x, y \in X$, $x \neq y$, and $2a \geq 2c = \|x-y\|$. The set*

$$E(x,y,a) = \{z \in X : ||z-x|| + ||z-y|| = 2a\}$$

is called the ellipse defined by its foci x *and* y.

DEFINITION 2.3.2 (based on a leading circle and one focus, [6]). *Let $L := (2a)\cdot K$ be a homothetic copy of the unit disk K, and $x \in L$ be an arbitrary point from it. The locus of points $z \in X$ for*

which there is a positive ε such that $z+\varepsilon K$ touches L and contains x on its boundary is called the ellipse defined by its leading circle and its focus x.

DEFINITION 2.3.3 (based on a leading line and a focus, [6]). *Let l be a straight line, x a point, and $\gamma = \frac{a}{c}$ a ratio larger than 1. The locus of points $z \in X$, for which there is a positive ε such that the boundary of the disk $z+\varepsilon K$ contains x and the disk $z+\gamma(\varepsilon K)$ touches the line l, is called* the ellipse defined by its leading line and its focus x.

The equivalence of these definitions for the Euclidean subcase is well known. We will prove that, while the first two definitions are equivalent also in normed planes, the third one yields a basically different class of curves.

PROPOSITION 2.3.1 ([6]). *In any normed plane the following holds: an ellipse, defined by its foci, is always an ellipse defined by its leading circle and a focus, and the converse statement is also true. On the other hand, an ellipse defined by its leading line and a focus is not necessarily an ellipse defined by its foci, and again the converse is true.*

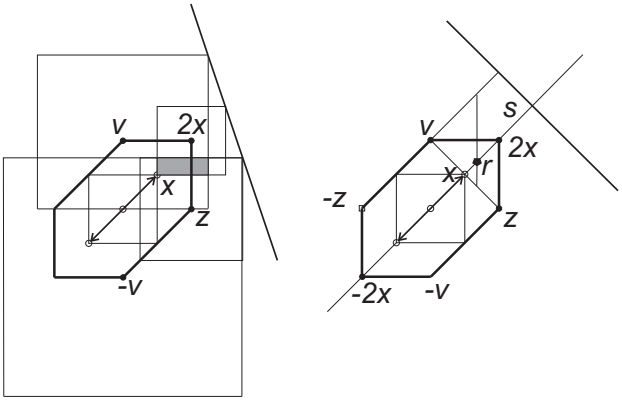

FIGURE 2.13. A metric ellipse which has no leading line

In Fig. 2.12 we can see that there is an ellipse following the third definition which is not centrally symmetric. By Theorem 2 of [146] it is not an ellipse by the first definition. Conversely, consider the ellipse $E(-x, x, 2)$ defined by its foci and shown in Fig. 2.13. First we can see that if it is also an ellipse defined by its leading line, then the leading line l and the new focus x' have to be in "symmetric position" with respect to the line joining the original foci. "Symmetric" means that this line is parallel to a diagonal of the unit square. In fact, if this is not the case, we get a figure as shown on the left side of Fig. 2.13. The squares S_{2x}, S_v, S_z, S_{-v} with centers $2x$, v, z, $-v$, respectively, touch l. The focus has to lie in the shaded rectangle, as the common point of the boundaries of homothetic copies $2x+(c/a)S_{2x}$, $v+(c/a)S_v$ and $z+(c/a)S_z$ of such squares (with a homothety ratio smaller than 1). On the other hand, the boundary of the square $-v+(c/a)S_{-v}$ intersects the shaded rectangle in a segment parallel to that one in which it is intersected by $z+(c/a)S_z$. So it is impossible to give a good position for the focus x'.
We now assume that l and x' have symmetric position (see the right side of Fig. 2.13). If this holds and the Euclidean distance of l and $2x$ is s, and that of x' and x is r, then, using the fact that the points $2x$, $-2x$ and v have to lie on the new ellipse, we have the equalities $r/s = (4-r)/(4+s) = (2-r)/(1+s)$, implying that $s = 1$ and $r = 2/3$ and showing that $a/c/ = 2/3$. Thus the leading line and the focus are both determined. On the other hand, the point $-z$ is not on the obtained ellipse, since the required ratio for it is $(12-\sqrt{2})/12 \neq 2/3$.
The examination of the ellipse defined by its leading line and its focus is new thus the following theorem is fundamental.

THEOREM 2.3.1 ([6]). *In a normed plane, an ellipse defined by its leading line and its focus is a convex curve, which is strictly convex if and only if this normed plane is strictly convex.*

A Euclidean hyperbola satisfies the same metric relations as a Euclidean ellipse, only that now the ratio $\frac{a}{c}$ is smaller than 1. The asymptotes of the hyperbola have directions $\frac{\sqrt{c^2-a^2}}{a}$, and the leading line intersects the asymptotes in points of the great circle. We also have three possible metric definitions. These are

DEFINITION 2.3.4 ([6]). *Given two points x, y in a normed plane and a distance denoted by $2a > 0$. Then $H(x, y, a) = \{z \in X : |\|z - x\| - \|y - z\|| = 2a\}$ denotes* the hyperbola defined by its foci x and y. *If $y = -x$, then we use the notation $H(x, a)$ for it.*

DEFINITION 2.3.5 (based on leading circle and focus, [6]). *Let $L := (2a) \cdot K$ be a homothetic copy of the unit disk K, and $x \in X$ be an arbitrary point exterior to L. The locus of points $z \in X$ for which there is a positive ε such that $z + \varepsilon K$ touches L and contains x on its boundary will be called* the hyperbola defined by its leading circle and its focus x.

DEFINITION 2.3.6 (based on leading line and focus, [6]). *Let l be a straight line, x be a point, and $\gamma = \frac{a}{c}$ a ratio less than 1. The locus of points $z \in X$, for which there is a positive ε such that the boundary of the disk $z + \varepsilon K$ contains x and the disk $z + \gamma(\varepsilon K)$ touches the line l, will be called* the hyperbola defined by its leading line and its focus x.

The analogue of Theorem 1 from [146] is given by our

THEOREM 2.3.2 ([6]). *Let $x \in S$ be a point of the unit circle. Then we have:*
(i) $H(x, 0)$ is the bisector corresponding to the vector x,
(ii) if there is a neighborhood of x on S in which S is strictly convex, then $H(x, 2)$ is the union of the two half-lines $[x, \infty)$ and $[-x, -\infty)$. If x is a point of a piecewise linear part of S, then it is the union of two closed cones.

The first statement is obviously true by the definition of the bisector given in the introduction. The second one follows from the concept and properties of d-segments in a Minkowski plane and from our definition of hyperbola; see [117], [116], and [28].
From the above theorem it can be seen that a connected part of $H(x, a)$ is, in general, not the boundary of a convex domain, because this property does not hold for a bisector; see [1] and [2].

THEOREM 2.3.3 ([6]). *The following two statements are equivalent to each other:*
(i) K is strictly convex.
(ii) For every $x \in S$ and for each value $a \in \mathbb{R}^+$ the set $H(x, a)$ is the union of two simple curves, each of which intersects any line parallel to $[-x, x]$ in precisely two points.

REMARK 2.3.1. *From the proof of this theorem we can conclude that the topological properties of hyperbolas do not depend on the parameter a and only on the position of their foci. Thus (ii) is equivalent to*
(iii) For every $x \in S$ there is a value $a \in \mathbb{R}^+ \cup \{0\}$ such that the set $H(x, a)$ is the union of two simple curves, intersected by any line parallel to $[-x, x]$ in precisely two points.

As in the case of ellipse we also have a proposition

PROPOSITION 2.3.2 ([6]). *In normed planes, a hyperbola defined by its foci is always a hyperbola defined by its leading circle and a focus. The converse statement is also true. In general, the third definition yields a different class of curves.*

On the base of this proposition the new curve the hyperbola defined by its leading line and we have a theorem on it, too.

THEOREM 2.3.4 ([6]). *The hyperbola defined by its leading line is the union of two simple curves. If the normed plane is strictly convex, then these curves cannot contain segments.*

For the case of parabolas, the first two definitions have no analogue, and so we had only the third case.

DEFINITION 2.3.7 ([6]). *In a normed plane, let l be a straight line, and x be a point. The locus of the points $z \in S$ for which there is a positive ε such that the boundary of the disk $z + \varepsilon K$ contains x and touches the line l, will be called* the parabola defined by its leading line and its focus x.

We also investigated the metric parabola and proved the theorem:

THEOREM 2.3.5 ([6]). *In a normed plane, the metric parabola is a simple curve which does not contain segments if and only if the normed plane under consideration is strictly convex.*

2.3.2. Roulettes (Common work with V. Balestro and H. Martini). We considered another important type of constructive curves in Minkowski plane, the so-called roulettes. In this part of the section we write capital letters like $A, B, \ldots$ for *points* with respective *position vectors* $a, b, \ldots$; by $a, b, \ldots, g(A, B)$ we denote *lines*, in the latter case spanned by A and B, and by AB the *segment* with endpoints A and B is meant. We use $\overrightarrow{AB}$ for the *vector* from A to B, or for the *half-line* starting at A and passing through B; sometimes we use also $a, b, \ldots r_1, r_2$ for half-lines (the respective meaning will be clear by the context). Further on, we write $\|a\|, \|a\|_E$ for the *general Minkowskian* and the *Euclidean norm* of a, respectively, and a^o stands for the *Minkowskian unit vector* parallel to a; $[a, b]$ is the *semi inner product* corresponding to the Minkowskian norm $\|\cdot\|$. Referring to the Minkowskian arc-length s, we denote by $r(s)$ the *radial function* of the Minkowskian unit circle, and by $\gamma(s)$ a *planar curve*, both parametrized by s; $\chi_\gamma(s)$ is the *Busemann curvature function* of $\gamma(s)$. The *Busemann sigma function* of the r-dimensional affine subspace V_r is $\sigma(V_r)$, and $(a, b)\angle$ denotes the *angle* determined by the lines a, b.

2.3.2.1. *Angle measures and general rotations.* The question how to measure angles is old and interesting. A good review of the history can be found in [20].

In [34], Busemann discussed the "axiom" for angle measures in the case of plane curves belonging to a class $\mathcal{S}$ of open Jordan curves, holding the additional property that any two distinct points lie on exactly one curve of $\mathcal{S}$. He defined the concepts of *ray* r, *angle* D with *legs* r_1 and r_2, and *angle measure* $|D|$ on the set of angles having the following properties:

(1) $|D| \geq 0$ (positivity),
(2) $|D| = \pi$ if and only if D is straight,
(3) if D_1 and D_2 are two angles with a common leg but with no other common ray, then $|D_1 \cup D_2| = |D_1| + |D_2|$ (additivity),
(4) if $D_\nu \to D$, then $|D_\nu| \to D$ (continuity).

He showed that these assumptions are sufficient to obtain many of the usual relationships between angle measure and curvature. We note that Busemann collected the essential properties of an angle measure that we have to require in every structure, where a natural concept of angle exists.

Lippmann [104] considered the classical Minkowski space defined on the n-dimensional Euclidean space by a "metrische Grundfunction" F, which is a positive, convex functional on the space being homogeneous of first degree. In our terminology, F is the *norm-square function* (a generalization of this concept can be found in this dissertation and in [8]. To have convexity (following Minkowski's definition), Lippmann required continuity of the second partial derivative, and positivity of the second derivative of F. Hence the unit ball of the corresponding space is always smooth. He used the arcus cosine of the bivariate function $(x, y) := (\sum x_i \frac{\partial}{\partial x_i} F(y))/F(x)$ to measure the angle between x and y. This yields a concept of transversality, namely: x is *transversal* to y if $(x, y) = 0$. A wide variety of angle measures referring to metric properties can be found in the literature. E.g., Lippmann's papers [105, 106] contain typically metric definitions of angle measures. For the situation in (normed or) Minkowski planes see, in addition to the papers already mentioned, Graham, Witsenhausen and Zassenhaus [69]. This paper refers to a useful metrical classification of angles by their measures, and a good review on this topic can be found in the book of Thompson [140].

In the last few decades some authors rediscovered this interesting problem in connection with the problem of orthogonality. We have to mention P. Brass who in [30] redefined the concept of angle measure as follows.

DEFINITION 2.3.8. *By an angle measure we mean a measure μ on the unit circle ∂B with center O which is extended in the usual translation-invariant way to measure angles elsewhere, and which has the following properties:*

(1) $\mu(\partial B) = 2\pi$,
(2) for any Borel set $S \subset \partial B$ we have $\mu(S) = \mu(-S)$, and
(3) for each $p \in \partial B$ we have $\mu(\{p\}) = 0$.

This concept was used in the papers of Düvelmeyer [48], Martini and Swanepoel [117], and Fankhänel [53, 54].
Another direction of research is to give immediate metric definitions of the angle of two vectors. In this direction we can find also papers of P. M. Miličič [121], C. R. Diminnie, E. Z. Andalafte, R. W. Freese [47] or H. Gunawan, J. Lindiarni and O. Neswan [79]. Further related papers on angle measures are [44], [45], [46], and [103].
As Busemann observed, the problem to find a natural definition of angular measure arises from the fact that the group of Minkowski rotations is very small. In a general normed space there are no such rotations which are also isometries of the space (see [62], [140], [113], and [114]). On the other hand, there are so-called left reflections (right-reflections) based on the notion of Birkhoff orthogonality (see [113] and [114]). These are not isometries, but they have some important properties of isometries; e.g., they are affine mappings of the plane sending lines into lines; the product of three left reflections in parallel lines in a strictly convex Minkowski plane is a left reflection in another line belonging to the same pencil of parallel lines; and the product of two left reflections in Birkhoff orthogonal lines is a symmetry of the plane. Unfortunately, if in a strictly convex and smooth Minkowski plane for left reflections the main lemma on three reflections with concurrent axes holds, then the plane is already Euclidean. Hence there is no chance to define an angle measure and also rotations by left reflections in the way that "a rotation is the product of two left reflections in non-parallel lines". This motivates our new definition of generalized angle measure and also the new concept of general Minkowski rotations, respectively.
In order to define a concept of rotation for a Minkowski plane, we start with extending the definition of Brass by considering Borel measures in a larger class of curves, not only in the unit circle, and we will derive angle measures for normed planes from it.

DEFINITION 2.3.9 ([7]). *Let $\gamma \subseteq X$ be a closed Jordan curve which is starlike with respect to a point p of the interior of the region bounded by γ. Let μ_γ be a (normalized) Borel measure on γ for which the following properties hold:*
(a) *$\mu_\gamma(\gamma) = 2\pi$;*
(b) *for any $q \in \gamma$ we have $\mu_\gamma(\{q\}) = 0$; and*
(c) *any non-degenerate arc of γ has positive measure.*
An angle measure defined in this way provides a translation invariant measure of angles *in the plane, which we define to be the convex hulls of two rays with the same starting point, or the half-plane given by two opposite rays. Given an angle $(r_1, r_2)\angle$ with apex a, we define its generalized angle measure $\mu_{\gamma,p}(r_1, r_2)$ to be the measure μ_γ of the arc determined on γ by the image of $(r_1, r_2)\angle$ via the translation $x \mapsto x - a + p$.*

Using this notion of generalized angle measure we define now the generalized rotations in Minkowski planes.

DEFINITION 2.3.10 ([7]). *Let $(X, ||\cdot||)$ be a Minkowski plane and let γ be a closed Jordan curve which is starlike with respect to a point p of the interior of the region bounded by γ. Let $\mu_{\gamma,p}$ be a generalized angle measure as in the previous definition. A general rotation (with respect to $\mu_{\gamma,p}$) is a transform $\mathrm{rot}_{\mu_{\gamma,p}} : X \to X$ for which the following three properties hold:*
(a) *The transform $\mathrm{rot}_{\mu_{\gamma,p}}$ leaves invariant the pencil $\mathcal{R}(p)$ of rays with origin in p. In other words, if $r \subseteq X$ is a ray with origin p, then $\mathrm{rot}_{\mu_{\gamma,p}}(r)$ is also a ray with origin p.*
(b) *For each $\alpha > 0$, $\mathrm{rot}_{\mu_{\gamma,p}}$ leaves invariant the homothetic curve $\gamma_{\alpha,p} := p + \alpha(\gamma - p)$, i.e., for such a curve we have $\mathrm{rot}_{\mu_{\gamma,p}}(\gamma_{\alpha,p}) \subseteq \gamma_{\alpha,p}$.*

(c) The function $r \in \mathcal{R}(p) \mapsto \mu_{\gamma,p}\left(\mathrm{rot}_{\mu_{\gamma,p}}(r), r\right)$ *is constant. Intuitively,* $\mathrm{rot}_{\mu_{\gamma,p}}$ *"rotates every ray of* $\mathcal{R}(p)$ *by a same angle".*

Notice that a general rotation can be considered as acting in the space of directions of X. Indeed, the set $\mathcal{R}(p)$ can be seen as this space. Later this viewpoint will be useful.
We emphasize that any general rotation relies on a fixed closed Jordan curve γ, an inner point p with respect to which γ is starlike, and a generalized angle measure $\mu_{\gamma,p}$. On the other hand, these three informations yield a certain class of general rotations, which we denote by $\mathcal{R}(\gamma, \mu, p)$. We head now to describe an element of such a class in terms of the angle of rotation. For any $\theta \in [0, 2\pi)$ we set $\mathrm{rot}_\theta : X \to X$ as follows: if $q_1 \in \gamma$, then q_1 is mapped to the (unique) point $q_2 \in \gamma$ taken counterclockwise, say, for which the rays $r_1 = [p, q_1\rangle$ and $r_2 = [p, q_2\rangle$ are such that $\mu(r_1, r_2) = \theta$. Now, any point $q \in X \setminus \gamma$ can be written in the form $q = p + \alpha\left(\mathrm{rad}_{\gamma,p}([p, q\rangle) - p\right)$ for some $\alpha \geq 0$, where $\mathrm{rad}_{\gamma,p} : \mathcal{R}(p) \to \gamma$ is the *radial function* which associates each ray starting at p to its intersection with γ. We just set

$$\mathrm{rot}_\theta(q) = p + \alpha\left(\mathrm{rot}_\theta\left(\mathrm{rad}_{\gamma,p}\left([p, q\rangle\right)\right) - p\right).$$

It is clear that $\mathcal{R}(\gamma, \mu, p) = \{\mathrm{rot}_\theta\}_{\theta \in [0,2\pi)}$. This description indicates that a class $\mathcal{R}(\gamma, \mu, p)$ has a group structure under composition, as in the standard Euclidean case. This is summarized in the following lemma.

LEMMA 2.3.1. *For a class* $\mathcal{R}(\gamma, \mu, p)$ *we have the following properties:*
(a) *Regarding composition,* $\mathcal{R}(\gamma, \mu, p)$ *is an abelian group. More precisely, we have* $\mathrm{rot}_{\theta_1} \circ \mathrm{rot}_{\theta_2} = \mathrm{rot}_{\theta_1 \oplus \theta_2}$*, where* $\oplus$ *is the sum modulo* 2π*.*
(b) *For any* $q \in \gamma$*, the application* $l \mapsto \mathrm{rot}_\theta(q)$ *is a bijection from* $[0, 2\pi)$ *to* γ*.*

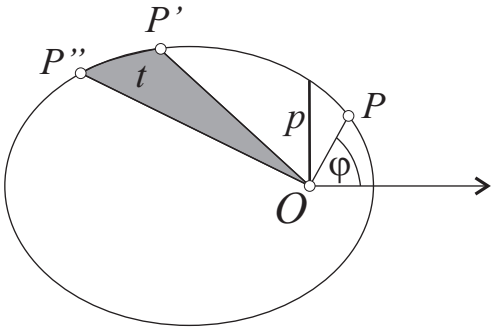

FIGURE 2.14. Area-based rotation

We highlight an interesting fact: The standard Euclidean rotation group can be obtained in any Minkowski plane. We just have to consider the group $\mathcal{R}(\gamma, \mu, o)$ where γ is the *Löwner ellipse*, which is defined as the ellipse of maximal volume contained in B, and μ is the measure given by twice the area of its sectors.
Next we give two examples of general rotations in the Euclidean plane. The first one relies on an area-based measure for an ellipse, which is clearly well defined.

EXAMPLE 2.3.1. Consider the Euclidean plane and the system of ellipses with common focus at the origin O and with major axis on the x-axis of the coordinate system, such that the positive half-line of x contains the closest point of the ellipse (see Fig. 2.14). In that polar coordinate system (which is called the heliocentric coordinate system for the ellipse), for which the ray $\varphi = 0$ is the positive half axis x, we can write the radial function $r(\varphi)$ of the ellipse G by the formula $r(\varphi) = p/(1 + \varepsilon \cos\varphi)$, where p is the semi-latus rectum of the ellipse and ε is the eccentricity of it, respectively. Let $\mu((\varphi', \varphi'')\angle)$ be the area of the sector enclosed by φ', φ'', and G be the arc between these lines. Hence

$$\mu((\varphi', \varphi'')\angle) = \frac{1}{2}\int_{\varphi'}^{\varphi''} \left(\frac{p}{1 + \varepsilon\,\cos\varphi}\right)^2 \mathrm{d}\varphi.$$

With respect to μ and G from above, for every real number $0 \leq t \leq 2\pi$ there is a generalized rotation of the Euclidean plane about O with this angle t. By Kepler's second law about planetary motions, the angle t of a generalized rotation is proportional to the time of the motion of the planet. Hence the generalized rotation with angle t maps the current position P' of the planet to that point P'' of the orbit where the planet arrives after time t. □

The principle of measuring the angle proportional to the area of the sector intersected by the angle domain from the basic disk $(G \cup \text{int}G)$ works in all Minkowski planes and for all basic curves G. Note that in the Euclidean plane with the unit circle as basic curve, this choice of μ gives the usual angle measure, and that we get the usual rotations as generalized rotations by choosing P to be the origin O. An advantage of this choice is affine invariance, but there is also a big disadvantage. Namely, the length of the arc G containing the domain of the angle cannot be calculated easily from this angle measure. (As a known example, we note that the calculation of the arc-length of an ellipse leads to a complete elliptic integral of second kind, which has no closed-form solution in terms of elementary functions.) In this paper we have to create tools for the so-called *rolling process*, which is a type of motion that combines rotation and translation of an object with respect to a given curve. More precisely, we combine two curves such that they are in contact with each other without sliding (no friction). Hence we have to compare the angle of rotations of the two curves by the fact that the swept arc-lengths do agree in the time of the moving. This requires a nice connection between the angle of the generalized rotation and the corresponding arc-length of the basic curve G.
The standard angle in the Euclidean plane can be obtained by considering arc-lengths in the unit circle, and hence the angle theory can be given in terms of the Euclidean norm. Of course, this can be carried over to Minkowski planes, and the general rotations given by the arc-length measure are possibly the most natural rotations in normed planes. We head now to take a better look at this particular case. We denote by l the Minkowski arc-length of a curve defined in the usual way: as the supremum of the sums of the lengths of the polygonal approximations of γ. Let $\gamma \in (X, ||\cdot||)$ be a closed rectifiable Jordan curve starlike with respect to an inner point p, and denote by μ_l the normalized Minkowski arc-length measure in γ. Formally, if $q_1, q_2 \in \gamma$, then

$$\mu_l(\text{arc}_\gamma(q_1, q_2)) = 2\pi \frac{l(\text{arc}_\gamma(q_1, q_2))}{l(\gamma)}.$$

Of course, μ_l is a generalized measure in the sense of Definition 2.3.9. Since the measure μ_l is induced by the geometry of the plane rather than being inherent to γ, one may wonder how the group $\mathcal{R}(\gamma, \mu_l, p)$ does rely on the initial γ and p that we have chosen. For example, in the Euclidean plane we can obtain the standard angle measure by considering the arc-length measure in any homothet of the unit circle and doing the usual normalization. Our next lemma shows that this is also true for arbitrary Minkowski planes.

LEMMA 2.3.2. *Let $\gamma \in X$ be a closed rectifiable Jordan curve starlike with respect to an inner point p, and let μ_l be the (normalized) Minkowskian arc-length measure. Given $\alpha > 0$, denote by $\gamma_{\alpha,p}$ the curve $p + \alpha(\gamma - p)$ homothetical to γ. Then $\mathcal{R}(\gamma, \mu_l, p) = \mathcal{R}(\gamma_{\alpha,p}, \mu_l, p)$.*

Despite having the good property shown above, the arc-length rotations are not at all linear transformations. For this reason we may face some difficulties when trying to derive closed formulas for them. But we have some exceptions. Next we give an example for the Minkowski arc-length rotation which coincides with an usual Euclidean rotation.

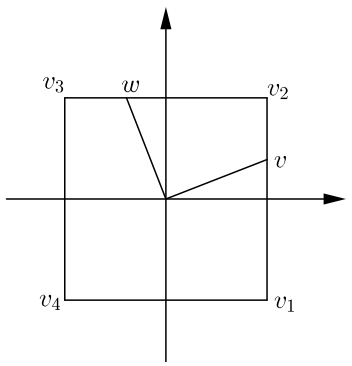

FIGURE 2.15. $\text{rot}_{\frac{\pi}{2}}(v) = w$

EXAMPLE 2.3.2. Consider the norm $||\cdot||_\infty$ defined in $\mathbb{R}^2$ to be $||(x, y)||_\infty = \max\{|x|, |y|\}$. The general rotation $\text{rot}_{\frac{\pi}{2}} : X \to X$ given by the Minkowski arc-length measure in the unit circle, and with respect to the origin, coincides with the usual Euclidean rotation of angle $\frac{\pi}{2}$. Indeed, the unit circle B of $(\mathbb{R}^2, ||\cdot||_\infty)$ is the square with vertices $\{(\pm 1, \pm 1)\}$ which, for the sake of simplicity of the used notation, we may denote in the counterclockwise way by v_1, v_2, v_3, and v_4. If $v \in [v_1, v_2]$, then $\text{rot}_{\frac{\pi}{2}}$ clearly maps v to the point w of the segment $[v_2, v_3]$ for which $||w - v_3|| = ||v - v_2||$ (see Figure 2.15). □

Translations are a simple kind of motion in Minkowski planes, and they are clearly isometries. The general rotations can also be seen as motions in the Minkowski plane, which are not necessarily isometries. Thus, we may consider the composition of translations and general rotations to obtain a larger class of motions in the Minkowski plane.

DEFINITION 2.3.11 ([7]). *Let $\mathcal{R}(\gamma, \mu, p)$ be a fixed group of general rotations, and for any $v, w \in X$ let $t_{vw} : X \to X$ denote the translation which maps v to w, i.e., $t_{vw}(x) = x - v + w$. We define the* motion group generated by $\mathcal{R}(\gamma, \mu, p)$ *to be the group of applications of the form $t_{pq} \circ \mathrm{rot} \circ t_{qp} : X \to X$, where $q \in X$ and $\mathrm{rot} \in \mathcal{R}(\gamma, \mu, p)$. When there is no possibility of confusion on the group of general rotations considered here, we will denote the motion group by $\mathcal{M}_r$.*

REMARK 2.3.2. *Notice that the motion group associated to $\mathcal{R}(\partial B, \mu_l, o)$, where μ_l is, as usual, the Minkowski arc-length measure, contains all direction-preserving isometries of the plane.*

2.3.2.2. *Motions of rigid systems in the Euclidean plane.* Consider a plane Σ' which is moving on the fixed plane Σ. The two simplest possibilities for such movements are given by translation and rotation. In Euclidean geometry we can substitute the planes with cartesian coordinate frames Oxy and $O'uv$. When we would like to describe the motion of a point P of the moving plane, we need the coordinates u, v of the point P in the moving frame, the coordinates p, q of O' in the fixed coordinate system, and the angle φ of the positive half of the X-axis of the fixed frame with the positive half of the x-axis of the moving frame. We get the coordinates x, y of the point P in the fixed system by

$$x = p + u\,\cos\varphi - v\,\sin\varphi\,, \qquad y = q + u\,\sin\varphi + v\,\cos\varphi.$$

Here p, q, φ are functions of a quantity t which determines the motion. (For example, t can denote the time, or any other metric parameter.) Assume that $\varphi(t)$ is not zero on an interval of t. Then it can be inverted, and p, q can also be considered as a function of φ. (This assumption says that our motion cannot contain translations in that domain. We call such a motion *non-translative planar motion*.) The derivative of the coordinate functions with respect to φ gives the coordinates of the velocity vector of the point P. It is more convenient to use vector equality, and hence we introduce some further notion. Let

$$\mathrm{R}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

denote the rotation about the origin with signed angle φ. Then the first equation array has the form $x = p + \mathrm{R}(\varphi)u$. If $\mathrm{Q} = \mathrm{R}(\pi/2)$ denotes the rotation with $\pi/2$, we have the following rules:

$$\mathrm{Q}^2 = -\mathrm{E}, \quad \mathrm{Q}^3 = \mathrm{Q}^{-1} = \overline{\mathrm{Q}} = -\mathrm{Q}, \quad \mathrm{Q}^4 = \mathrm{E},$$

where E is the unit matrix. We denote by "$\dot{\cdot}$" the *derivative with respect to* φ, which means in this section the Euclidean arc-length parameter. It is clear that $\dot{\mathrm{R}} = \mathrm{QR}$ and thus $\dot{(\mathrm{R}^{-1})} = -\mathrm{QR}^{-1}$. For every value of φ there is precisely one point u_0 of the moving plane for which the velocity vector vanishes. This is

$$u_0 = \mathrm{QR}^{-1}\dot{p}.$$

This point u_0 of the moving plane is a so-called *instantaneous center* (or *instantaneous pole*) of the motion, and the set of these points is the *moving polode (centroid)*, or curve γ' of instantaneous poles, of the moving plane. The points of the moving polode can also be obtained as rest in the frame. These points x_0 are described by

$$x_0 = p + \mathrm{R}u_0 = p + \mathrm{Q}\dot{p}\,.$$

They form the so-called *fixed polode (centroid)*, or curve γ of instantaneous centers, in the fixed plane. We examine the motion with respect to the point x_0. If x is arbitrary, then $x - x_0 = \mathrm{R}u - q\dot{p}$, and using the equality $\dot{x} = \dot{p} + \mathrm{QR}u$, we have $\mathrm{Q}\dot{x} = \mathrm{Q}\dot{p} + \mathrm{QR}u$. Since $x - x_0 = \mathrm{R}u - \mathrm{Q}\dot{p}$, we get that

$$\dot{x} = \mathrm{Q}(x - x_0).$$

Hence the velocity vector of the motion at the point x is orthogonal to the position vector from x_0 to x. This implies that the moving system in the given moment is a rotation about the center x_0. Observe that the velocity vectors of the two polodes at their common point agree; in fact,

$$\dot{u}_0 = \mathrm{Q}\dot{\mathrm{R}^{-1}}\dot{p} = \mathrm{R}^{-1}\dot{p} + \mathrm{Q}\mathrm{R}^{-1}\ddot{p} = \dot{x}_0.$$

Hence the arc-length elements of the two curves agree, and we get that in every moment the two curves are touching. Also we see that their arc-lengths calculated from a point φ_0 to the point φ have the same value. Hence the moving polode γ' *rolls without slipping* (or without friction) on the fixed polode γ, and this is the only rolling process which corresponds to the given motion of the planes. Hence we see the fact *that every non-translatory planar motion of a rigid mechanical system in the plane can be considered as the rolling process of a curve rigidly connected with the system on a fixed curve in the plane.* This motivates the so-called main theorem of planar kinematics, namely

THEOREM 2.3.6 ([65]). *At every moment, any constrained non-translatory planar motion can be approximated (up to the first derivative) by an* instantaneous rotation. *The center of this rotation is called the* instantaneous pole. *Thus, for each position of the moving plane, we generally have exactly one point with velocity zero (as a result of that, the instantaneous pole is also called velocity center).*

This theorem leads to an interesting class of curves in the Euclidean plane.

DEFINITION 2.3.12 ([65]). *Given a curve γ' associated with a plane Σ' which is moving so that the curve rolls, without friction, along a given curve γ associated with a fixed plane Σ and occupying the same space. Then a point P attached to Σ' describes a curve in Σ called a* roulette.

Based on this rolling process we can rewrite the definition of the motion of rigid systems. Observe that every planar motion implies the motion of all points of the moving plane with respect to the fixed one. These orbits are said to be roulettes. Thus, for the studied motion we consider two curves, also called *polodes*, and a suitable rolling process to determine the motion of a singular point. For this purpose a method is needed to determine the fixed position of the point P with respect to the moving polode. A usual method is to give a line through the point P which intersects the moving polode in the point Q and fixes the distance of P and Q and the angle of the line PQ with the tangent line t_Q of the moving polode at Q. Hence the choice of Q on the moving polode is arbitrary. Fix $Q = w(0)$ and $P = x(0)$. The points of the roulette $w(s)$ of Q can be obtained by the composition of the following transformations: translate the point $\gamma'(s)$ into the origin, rotate the image of the point of $\gamma(0)$ about the origin by the angle $\varphi(s) = \left(\dot{\gamma}(s), \dot{\gamma'}(s)\right) \angle$, and translate the obtained point by $\gamma(s)$. Hence the roulette of Q in the fixed system is given by

$$w(s) = \mathrm{R}(\varphi(s))(-\gamma'(s)) + \gamma(s) = \gamma(s) - \mathrm{R}(\varphi(s))(\gamma'(s)).$$

Since the roulette $x(s)$ of the point P can be described by the formula $x(s) = w(s) + \mathrm{R}(\varphi(s))p$, we get

$$x(s) = \gamma(s) + \mathrm{R}(\varphi(s))\left(p - \gamma'(s)\right). \tag{21}$$

This means that if we have two touching arcs $\gamma(s)$ and $\gamma'(s)$ of a plane Σ, and we associate to the second arc a moving plane Σ' in which its position is fixed, then the rolling process of $\gamma'(s)$ on $\gamma(s)$ (locally) determines an orbit of every point of Σ' in a unique way. In the Euclidean plane, (21) shows that in every moment with respect to varying p we have an isometry. Hence the rolling process of the arcs determines a rigid motion of the plane Σ'. This representation is locally unique, since a rigid motion uniquely determines its polodes. Hence we have

THEOREM 2.3.7 ([7]). *If $\gamma, \gamma' : [0, \beta] \to \mathbb{R}^2$ are two simple Jordan arcs with common touching point $\gamma(0) = \gamma'(0)$ such that s is the arc-length parameter of both of them (considered from the*

points $\gamma(0), \gamma'(0)$ to the points $\gamma(s)$, $\gamma'(s)$, respectively), then for every $s \in [0, \beta]$ we have an isometry Φ_s sending the original position vector p into the instantaneously position $\Phi_s(p)$. If γ and γ' have, for all $s \in [0, \beta]$, unique tangents at their points $\gamma(s)$ and $\gamma'(s)$, respectively, then, for all $s \in [0, \beta]$, Φ_s is uniquely determined and can be described by the vector equation

$$\Phi_s(p) = \gamma(s) + \mathrm{R}\left(\left(\dot{\gamma}(s), \dot{\gamma}'(s)\right) \angle\right)(p - \gamma'(s)) \ .$$

Here $\dot{\gamma}(s)$ and $\dot{\gamma}'(s)$ denote the unit tangent vectors at $\gamma(s)$ and $\gamma'(s)$, respectively, and $\mathrm{R}(\theta)$ is the rotation with the angle θ. For fixed p, the graph of the function $\Phi_{(.)}(p) : [0, \beta] \to \Sigma$ is said to be the roulette *of the point $P = p \in \Sigma$ for the rigid motion given by the system of isometries $\{\Phi_s : s \in [0, \beta]\}$.*

2.3.2.3. *Flexible motions of a Minkowski plane.* Our purpose now is to extend Theorem 2.3.7 to Minkowski planes. For this purpose we defined already the motion group $\mathcal{M}_r$ of the Minkowski plane, which is a good analogue of a motion group of the Euclidean plane. Clearly, we have to omit the condition that a motion is an isometry, due to the smallness of the actual isometry group in a Minkowski plane. Of course, any motion group $\mathcal{M}_r$ contains all the translations. On the other hand, it is possible that the image of a metrical segment under a general rotation is not a metrical segment. Hence the concept of Euclidean rigid motions has to be redefined. This is not a strange project because of in practice there is no rigid motion. To a plausible example consider the rolling process of a wheel of a car. Since the tyre continuously change its shape to a good modelling of this motion we should omit the requirement that the motion is rigid. (See Fig 2.16.)

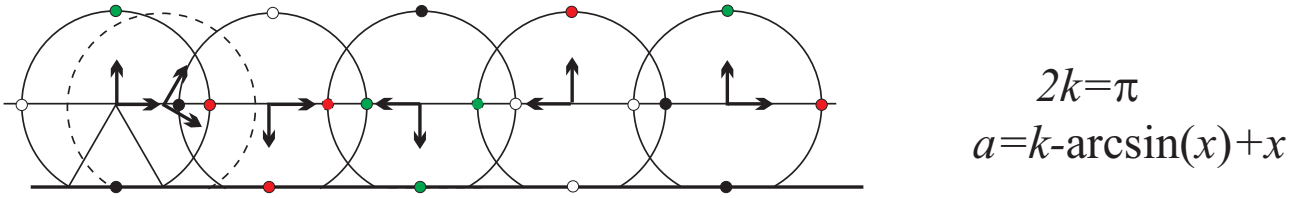

FIGURE 2.16. Motion of a wheel. The arc-lengths between the points labelled with circles are changing, continuously.

We concentrate on Theorem 2.3.7 for the Euclidean planar motions, and we will consider from now on that the motion group $\mathcal{M}_r$ is the motion group associated with the group of general rotations $\mathcal{R}(\partial B, \mu_l, o)$. In other words, we will consider the rotations by arc-length of the unit circle with respect to the origin.

DEFINITION 2.3.13. *The rectifiable Jordan curve $\gamma'(s)$* rolls without slipping *on the rectifiable Jordan curve $\gamma(s)$ if in every moment $s \in [0, \beta]$ the two curves touch each other, and the respective arc-lengths calculated from their common point $\gamma(0) = \gamma'(0)$ to the other one $\gamma(s) = \gamma'(s)$ are equal to each other and also to the common parameter s.*

Having the rolling procedure and the motion group $\mathcal{M}_r$, we can define the continuous (but not rigid) motions of a Minkowski plane. Assume that in this section any considered curve is a rectifiable Jordan curve, with unique tangent at all of its points, respectively. We denote the unit tangent vector of γ at its point $\gamma(s)$ by $\dot{\gamma}(s)$. (Since s means the arc-length parameter, this notation corresponds to the usual Euclidean notation based on the arc-length derivative of the position vector.)

DEFINITION 2.3.14. *If the rectifiable Jordan curve $\gamma'(s)$ rolls, without slipping, on the rectifiable Jordan curve $\gamma(s)$, then we define* the flexible motion corresponding to the rolling curves γ *and γ' as the following set of mappings:*

$$\{\Phi_s(p) = \gamma(s) + \mathrm{R}(\varphi_s)\,(p - \gamma'(s)) : \ s \in [0, \beta]\},$$

where $\mathrm{R}(\varphi_s) \in \mathcal{R}(\partial B, \mu_l, o)$ *denotes the general rotation which maps the (oriented) direction* $\dot\gamma(s)$ *to the (also oriented) direction* $\dot\gamma'(s)$. *A curve given by the graph of a fixed point* $p = P$ *is called the* roulette *of* P.

The vector $\frac{\partial \mathrm{R}(\varphi)}{\partial \varphi}(x) = \lim_{\varepsilon\to 0} \frac{\mathrm{R}(\varphi+\varepsilon)(x)-\mathrm{R}(\varphi)(x)}{\varepsilon}$ is the tangent vector of $|x|\partial B$ at the point $\mathbf{x}$. This means that $\frac{\partial \mathrm{R}(\varphi)}{\partial \varphi}(x)$ is Birkhoff normal to the vector $\mathrm{R}(\varphi)(x)$. (For relations between semi inner products and Birkhoff orthogonality, see, e.g., [8] or [9].) Denote by Q that mapping which sends the vectors to their Birkhoff normals with the same norm, and by Q^{-1} the mapping which sends the vectors to their Birkhoff transversals with the same lengths. (Note that Birkhoff orthogonality is not a symmetric relation; see, e.g., [115] or [119]. So, in general, if x is Birkhoff normal to y, then y not to x. However, we have a possibility to "reverse" the formulation "x is Birkhoff normal to y". We say in this case that y is transversal to x.) Since the tangent vector of the roulette of P at the point with parameter s is

$$\dot\Phi_s(p) = \dot\gamma(s) + \mathrm{Q}(\mathrm{R}(\varphi(s))(p-\gamma'(s))\dot\varphi(s) - \mathrm{R}(\varphi_s)\dot\gamma'(s) = \mathrm{Q}(\mathrm{R}(\varphi(s))(p-\gamma'(s))\dot\varphi(s),$$

we get that $\left[\dot\Phi_s(p), \Phi_s(p)-\gamma(s)\right] = 0$. Hence we obtain

STATEMENT 2.3.1 ([7]). *The velocity vector of the flexible motion of a point* $\Phi_s(p)$ *of the roulette in a moment* s *is Birkhoff normal to that vector* $\Phi_s(p)-\gamma(s)$ *which shows from the point to the instantaneous pole of the motion.*

From Statement 2.3.1 we can see that our definition yields the same kinematics in the Minkowski plane as given by usual motions of rigid systems in the Euclidean plane.

2.3.2.4. *Curvature and the Euler-Savary equations.* We proved the so-called *Euler-Savary equations* (see [129]) for normed planes. In space-time this was investigated by several authors (e.g. Ikawa [91], [51], [52]). Ikawa defined roulettes and proved the Euler-Savary equations for normed planes, with respect to this semi-Riemannian geometry of constant curvature. Because of the rich isometry group of this plane, the validity of these results is not so surprising as in our case.

In this section we have to assume second order differentiability of the unit circle, and we have to introduce the concepts of curvature and curvature radius of a curve, respectively. Fortunately, in Minkowski planes several such concepts are known. Curvatures for curves in Finsler spaces were introduced for dimension $n = 2$ by Underhill [142] and Landsberg [100]. For general n they were introduced by Finsler [59, 60]. The definitions coincide for $n = 2$. The underlying idea of these definitions is this: If $\gamma(s)$ is a curve with tangent t at a given point q, then the line parallel to this tangent through the origin intersects the unit circle in a point q' (in fact, in a pair of points, but it will not matter which point is chosen). There is exactly one ellipsoid with the origin as center through q' which has at q' the same second differential as the unit circle. This ellipsoid determines a Euclidean metric $E(q)$. Finsler defines the curvatures of $\gamma(s)$ at q as the curvatures at q of $\gamma(s)$ as a curve in $E(q)$. Obviously, $E(q)$ exists only if the unit circle has a second differential at q' and the indicatrix is a non-degenerate ellipse. Actually, this idea is significant only if C is of class C^2 and has positive Gauss curvature. Thus $\gamma(s)$ may not even have a curvature when it is analytic.

There exists another definition of curvature for curves in general spaces which is due to Menger [120] (for modifications of this concept see [87]). Haantjes' curvature coincides with that of Finsler. Hence Haantjes' main result in [87] means that, in Minkowski spaces, Menger's definition coincides with Finsler's definition.

In [36], Busemann gave another concept of curvature[3].

[3]There is a nice connection between the concepts of curvature given by Finsler and Busemann. In a Minkowski plane, the Finsler curvature χ^f and the curvature χ of Busemann of a curve $\gamma(s)$ at a point P, with position vector $\overline{p}$, are related by

$$(\chi^f(P))^2 = \frac{\chi^2(P)}{\chi_T(\overline{p})},$$

In n-dimensional Minkowski space let $\gamma(s)$ be a curve which is, in the Euclidean sense, of class C^r and parametrized by the Minkowskian arc-length s. Let $\gamma(s_i)$, $i = 0, 1, \ldots, n$, be $n+1$ points on $\gamma(s)$. Let T_r denote the r-dimensional Minkowski volume of the r-dimensional simplex that is spanned by the points $\gamma(s_i)$, $i = 0, 1 \ldots r$. Then we define the $(r-1)$-*th curvature* χ_{r-1} of the curve γ in its point $\gamma(s)$ by the limit

$$\chi_{r-1}(s) = \frac{r^2}{r-1} \lim_{s_i \to s} \frac{1}{\|\gamma(s_r) - \gamma(s_0)\|} \frac{T_r T_{r-2}}{T_{r-1} T^\star_{r-1}}$$

(see [36]), where $T^\star_{r-1}$ denotes the volume of the $(r-1)$-dimensional simplex spanned by the points $\gamma(s_i)$, $i = 1, \ldots r$. Let D_r be the following quantity:

$$D_r(s) = r! \prod_{i=1}^{r} i! \lim_{s_i \to s} \frac{T_r}{\prod_{i<j} \|\gamma(s_i) - \gamma(s_j)\|}.$$

Then for $D_{r-2}(s) \neq 0$ we get the following form of the curvature function:

$$\chi_{r-1}(s) = \frac{D_r(s) D_{r-2}(s)}{D^2_{r-1}(s)}.$$

This formula can be rewritten by the concept of the general sine function of two flats of the n-dimensional Minkowski space, but we need only the case of dimension 2. Hence, using that $D_0(s) = 1$, the curvature is

$$\chi_\gamma(s) := \chi_1(s) = \frac{D_2(s)}{D^2_1(s)} = 2 \lim_{s_0, s_1, s_2 \to s} \frac{\mathrm{sm}(g(\gamma(s_0), \gamma(s_1)), g(\gamma(s_1), \gamma(s_2)))}{\|\gamma(s_2) - \gamma(s_0)\|},$$

where $g(x, y)$ denotes the line through x and y.
A curve $\gamma(s)$ having curvature in Euclidean sense has also curvature in the sense of Busemann. These two curvatures can be compared. For this purpose we have to use the σ-function introduced by Busemann. Let V_r be an r-flat of a Minkowski space of dimension n. If $U(V_r)$ is the set in which the r-flat, parallel to V_r and passing through the origin, intersects the solid Minkowskian unit sphere, then we define $\sigma(V_r)$ as the ratio of the r-dimensional volume of the r-dimensional unit ball and the Euclidean volume of $U(V_r)$. Observe that if $\gamma(s)$ is a C^1 curve with tangent line t_P and velocity vector $\dot\gamma(s)$ at the point $P = \gamma(s)$, then by the definition of Minkowski length we have $\|\dot\gamma(s)\| = \sigma(t_P)\|\dot\gamma(s)\|_E$, where $\|\cdot\|_E$ means the Euclidean norm. Busemann [36] proved that if $\chi_E(P)$ denotes the Euclidean curvature of $\gamma(s)$ at the point P, t_P is written for the tangent line of $\gamma(s)$ at P, and T_P is the osculating plane of the curve at P, then

$$\chi(P) = \frac{\sigma(T_P)}{\sigma^3(t_P)} \chi^E(P).$$

We use these formulas to establish a close analogue to the Euler-Savary theorem on rigid motions in the Euclidean plane. First of all, we consider two curves γ and γ'. Hence we have to use a suitable lower subscript for the curvature function. We also have the concept of *curvature radius* r_γ which is, as well-known, the reciprocal value of the curvature at the given point $K = \gamma(s)$. With these notions we are able to formulate

THEOREM 2.3.8 (Second Euler-Savary equation). *If the unit circle of the Minkowski plane is two times continuously differentiable, then the following equality holds:*

$$\chi_\gamma - \chi_{\gamma'} = \frac{1}{r_\gamma} - \frac{1}{r_{\gamma'}} = \frac{\sigma(T_K)}{\sigma^2(t_K)} \frac{1}{\alpha_K}. \tag{22}$$

Here r_γ is the curvature radius of the fixed polode at its point $K = \gamma_s$, $r_{\gamma'}$ is the curvature radius of the moving polode at its point $K = \gamma'_s$, and α_K is the length of the common velocity vector of the fixed and moving polodes at the moment s and at the instantaneous pole $K = \gamma(s) = \gamma'(s)$.

where $\chi_T(\overline{p})$ is the curvature of the isoperimetrix (see [35]) at a point $\overline{p}$ (the tangent of the isoperimetrix has to be parallel to the tangent of $\gamma(s)$ at p).

To prove an analogue of the first Euler-Savary equation, we need a deeper investigation of the Busemann curvature. Let t_K be the common tangent of the polodes at their common point K, which is the x-axis of a Euclidean orthogonal coordinate system (x, y). We denote by O, O' the curvature centers of the curves $\gamma(s)$ and $\gamma'(s)$, respectively. Then O and O' coincide with the line y and $\chi^E_\gamma(K) = 1/\|KO\|_E$, $\chi^E_{\gamma'}(K) = 1/\|KO'\|_E$. Denote by P any point of the moving plane corresponding to the curve γ' with the vector $p = \overrightarrow{KP}$. As we saw in Statement 2.3.1, the line n_P of the points K, P contains the Minkowskian curvature center of the roulette of P, since it is Birkhoff normal to the tangent t_P at P. Denote this point by P'. We have at $\gamma(0) = \gamma'(0) = K$ that $\mathrm{R}(\varphi(0)) = \mathrm{id}$, and $\dot\gamma(0) = v_K$, where v_K is the common (Minkowskian) velocity vector at K. Hence we have the equality $v_P := \left.\frac{\partial(\Phi_s(p))}{\partial s}\right|_0 = \mathrm{Q}(\mathrm{R}(\varphi(s))(p - \gamma'(s)))\dot\varphi(s)|_0 = \mathrm{Q}(\overrightarrow{KP})\dot\varphi_0$. Thus, the acceleration vector a_P is

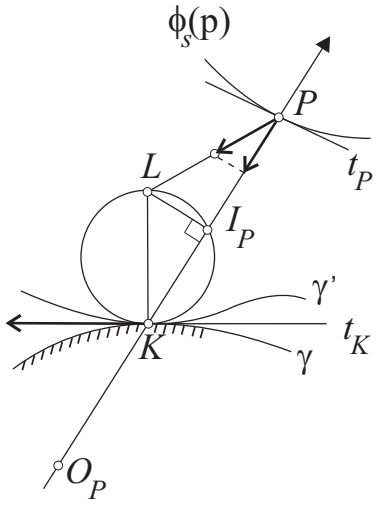

FIGURE 2.17. The point L

$$a_P = \left.\frac{\partial v_P}{\partial s}\right|_0 = \lim_{\varepsilon\to 0}\frac{\mathrm{Q}(\mathrm{R}(\varphi(\epsilon)))(p-\gamma'(\epsilon))\dot\varphi(\epsilon) - \mathrm{Q}(\mathrm{R}(\varphi(0))(p-\gamma'(0)))\dot\varphi(0)}{\varepsilon} + \mathrm{Q}(\overrightarrow{KP})\ddot\varphi(0) =$$

$$= \dot\varphi(0)\left(\lim_{\varepsilon\to 0}\frac{\mathrm{Q}(\mathrm{R}(\varphi(\epsilon)))(p-\gamma'(\epsilon)) - \mathrm{Q}(p-\gamma'(\epsilon))}{\varepsilon} + \lim_{\varepsilon\to 0}\frac{\mathrm{Q}(p-\gamma'(\epsilon)) - \mathrm{Q}(p-\gamma'(0))}{\varepsilon}\right) + \mathrm{Q}(\overrightarrow{KP})\ddot\varphi(0).$$

Observe that if Q would be an additive function and we could change it with the limit process, then the first term in the bracket could be simplified to the quantity $\mathrm{QQ}(\overrightarrow{KP})\dot\varphi(0)$ and the second one is nothing else than the velocity vector of the moving polode at zero. (In our case it is also the velocity vector of the fixed polode.) In general this is not so, since the additivity of the operation Q implies that the space is Euclidean with a standard inner product. Thus, for further investigations we need a quantity which measures the difference between the given limits and the optimal values (attended by the case of inner product planes). This motivates the following lemma.

LEMMA 2.3.3. [7] *Assume that $\gamma(s)$ is a curve of C^1 type parametrized by its arc-length. If $a, b, c \in \gamma(s)$ and t_c denotes the tangent of the curve $\gamma(s)$ at its point c, then we have*

$$\lim_{a,b\to c}\frac{\mathrm{Q}(b) - \mathrm{Q}(a)}{\|b-a\|} = \frac{1}{\sigma(t_c)}\mathrm{Q}^2(c).$$

By Lemma 2.3.3 we get an expression for the acceleration vector above, namely

$$a_P = \dot\varphi^2(0)\left(\frac{1}{\sigma(t_P)}\mathrm{Q}^2(\overrightarrow{KP}) - \frac{1}{\sigma(t_K)}\mathrm{Q}\left(\frac{v_K}{\dot\varphi(0)}\right)\right) + \mathrm{Q}(\overrightarrow{KP})\ddot\varphi(0),$$

where v_K means the common velocity vector of the curves $\gamma(s)$, $\gamma'(s)$ at $K = \gamma(0) = \gamma'(0)$. We now introduce a point L (see Figure 2.17) such that

$$\overrightarrow{LP} = -\left(\frac{1}{\sigma(t_P)}\mathrm{Q}^2(\overrightarrow{KP}) - \frac{1}{\sigma(t_K)}\mathrm{Q}\left(\frac{v_K}{\dot\varphi(0)}\right)\right),$$

hence the acceleration may be written as $a_P = \ddot\varphi(0)\mathrm{Q}(\overrightarrow{KP}) - \dot\varphi^2(0)\overrightarrow{LP}$. Observe that $\mathrm{Q}(\overrightarrow{KP})$ is normal to the vector $\overrightarrow{KP}$, and that it has no component parallel to $\overrightarrow{KP}$. The vector $-\dot\varphi^2(0)\overrightarrow{LP}$ lies along $g(L, P)$ and is directed toward L, so its projection contributes to both components (one of them parallel to $g(K, P)$, and the other one normal to it) of the acceleration vector. Hence a unique situation exists if $\overrightarrow{LP}$ is normal to $\overrightarrow{KP}$. In this case, the acceleration vector has no component parallel to $g(K, P)$ implying that the radius of curvature of its path is infinite.

DEFINITION 2.3.15. [7] *The locus of all points P for which $\overrightarrow{LP}$ is normal to $\overrightarrow{KP}$ is the* inflection curve *of the motion. The point L is the* inflection pole *of the motion.*

The inflection curve is the "Thales circle" of the segment $\overline{KL}$ with respect to Birkhoff orthogonality. We have to prove the following properties of it:

STATEMENT 2.3.2. [7] *The inflection curve ι is a closed curve. It is starlike with respect to the point K if the unit circle is smooth. However, in general it does not bound a convex domain. Finally, if it is a Minkowski circle for all segments (at least one segment) of the normed plane, then the plane is Euclidean.*

To prove the starlike property, consider the notation of Fig. 2.18.

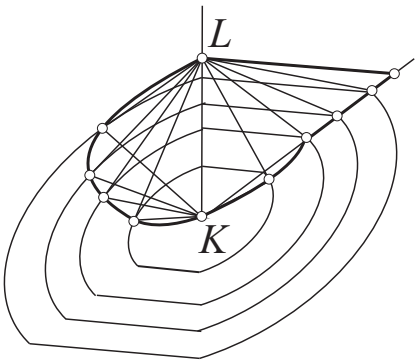

FIGURE 2.18. The curve of inflection

By the physical meaning of the acceleration vector, the absolute value of the normal component of this vector is $\dot{\varphi}^2(0)\|\overrightarrow{KP}\|^2\chi(P) = \dot{\varphi}^2(0)\frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{PO_P}\|}$, where $\chi(P)$ and $\|\overrightarrow{PO_P}\|$ are the curvature and the curvature radius R_P of the roulette at P, respectively. Along the path, the direction is always normal. If this normal is oriented from K to P, then the magnitude and orientation of the normal component of the acceleration vector may be defined in terms of real numbers, and it will be positive if PO_P is positive, i.e., if it has the same orientation as KP. If PO_P has orientation opposite to that of KP, it will be negative.

On the other hand, it can also be obtained from the length of the orthogonal projection of $\dot{\varphi}^2(0)\overrightarrow{PL}$ to the path normal line $g(P,K)$. Hence we have

$$\dot{\varphi}^2(0)\frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{PO_P}\|} = \dot{\varphi}^2(0)\left[\frac{1}{\sigma(t_P)}\mathrm{Q}^2(\overrightarrow{KP}) - \frac{1}{\sigma(t_K)}\mathrm{Q}\left(\frac{v_K}{\dot{\varphi}(0)}\right), (\overrightarrow{KP})^0\right],$$

with $(\overrightarrow{KP})^0$ as unit vector. Denote the second intersection point of the line $g(K,P)$ with the inflection curve by I_P. Then

$$\overrightarrow{PI_P} = \frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{PO_P}\|}(\overrightarrow{KP})^0 = \left[\frac{1}{\sigma(t_P)}\mathrm{Q}^2(\overrightarrow{KP}) - \frac{1}{\sigma(t_K)}\mathrm{Q}\left(\frac{v_K}{\dot{\varphi}(0)}\right), (\overrightarrow{KP})^0\right](\overrightarrow{KP})^0,$$

and so we have the equality

$$\frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{O_PP}\|} = \|\overrightarrow{I_PP}\|.$$

Hence we get the following geometric form of the first Euler-Savary theorem.

THEOREM 2.3.9. [7] *The instantaneous center K and the curvature center O_P of the roulette at its point $P \neq K$ satisfy the equality*

$$\|\overrightarrow{O_PP}\| = \frac{\|\overrightarrow{KP}\|^2}{\|\overrightarrow{I_PP}\|},$$

where the second intersection point of the path normal line at P with the inflection curve is the point I_P.

By the law of sine introduced earlier, O_PP and I_PP are always marked off in the same orientation along the line KP. Thus, when I_P has been established, the orientation of I_PP gives the orientation of O_PP. Hence equality above has an equivalent form for directed segments (with Minkowski lengths):

$$\frac{1}{KP} - \frac{1}{KO_P} = \frac{1}{KI_P}.$$

From this equality we can see immediately that the curvature radius of the point of the inflection curve is infinite. Similarly, the centers of path curvature of all points at infinity are on the *return*

curve obtained as the image of the inflection curve under reflection at the point K. To see a connection between the two Euler-Savary equations, we give a connection between KI_P and α_K which is the length of the common velocity vector of the fixed and moving polodes at K. Before discussing it, we define Busemann's sine function $\mathrm{sm} : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$ from the pairs of lines to the field of reals. If $a, b \in \mathcal{L}$ and s_a, s_b are two segments on these lines, respectively, then we can define the parallelogram $\pi(s_a, s_b)$ that is spanned by s_a and s_b. If we write $\mathrm{area}(\pi(s_a, s_b))$ for the Busemann area of $\pi(s_a, s_b)$ and take into consideration the Minkowski lengths $|s_a|$, $|s_b|$ of s_a and s_b, then the *Minkowski sine function of Busemann* can be defined as follows:

$$\mathrm{sm}(a, b) := \frac{\mathrm{area}(\pi(s_a, s_b))}{\|s_a\|\|s_b\|}. \tag{23}$$

From the definitions of Minkowski length and Minkowski area it follows that $sm(a, b)$ is not depending on the segments s_a and s_b. Thus, it depends only on the lines a, b. For the sine function $\mathrm{sm}(g_1, g_2)$ of Busemann the theorem of sines holds, and it is compatible with the normality concept of Birkhoff. Hence we have

$$\frac{\|\overrightarrow{KI_P}\|}{\|\overrightarrow{KL}\|} = \frac{\mathrm{sm}(g(K,L), g(L,I_P))}{\mathrm{sm}(g(K,I_P), g(L,I_P))} = \frac{\sin(g(K,L), g(L,I_P))\frac{\sigma(T_K)}{\sigma(g(K,L))\sigma(g(I_P,L))}}{\sin(g(K,I_P), g(L,I_P))\frac{\sigma(T_K)}{\sigma(g(K,I_P))\sigma(g(I_P,L))}} = \sin\Psi\frac{\sigma(g(K,P))}{\sigma(g(K,L))},$$

where Ψ is the Euclidean angle between the tangent line t_K at K and the line $g(K, P)$. From this we get the common form of the first and second Euler-Savary equations. By

$$\left(\frac{1}{KP} - \frac{1}{KO_P}\right)\mathrm{sm}(g(K,P), t_K)\frac{\sigma(t_K)\sigma(g(K,P))}{\sigma(T_K)} = \left(\frac{1}{KP} - \frac{1}{KO_P}\right)\sin\Psi = \frac{\sigma(g(K,L))}{\sigma(g(K,P))}\frac{1}{KL},$$

and using that the velocity vector v_K of the instantaneous pole at K is equal to $V_K = \dot{s}(0)\left.\frac{\partial\gamma(s(\omega))}{\partial s}\right|_0 = \alpha_K v_K^0$, we get that the acceleration vector is $a_K = \ddot{s}(0)v_K^0 + \alpha_K n_K^0$. This implies that its normal component is $[n_K^0, a_K]n_K^0 = \alpha_K n_K^0$. On the other hand, from the definition of the point L and the continuity property of the examined curves we get that if P tends to K, then $\overrightarrow{LP}$ tends to

$$\overrightarrow{LK} = \frac{1}{\sigma(t_K)}\mathrm{Q}\left(\frac{v_K}{\dot{\varphi}(0)}\right).$$

So we have $\|\overrightarrow{LK}\| = \alpha_K / (\sigma(t_K)\dot{\varphi}(0))$, and if we assume that the length of the directed segment KL is positive, then we get

$$\left(\frac{1}{KP} - \frac{1}{KO_P}\right)\mathrm{sm}(g(K,P), t_K)\frac{\sigma(t_K)\sigma^2(g(K,P))}{\sigma(T_K)\sigma(g(K,L))} = \frac{1}{\|\overrightarrow{KL}\|} = \frac{\sigma(t_K)\dot{\varphi}(0)}{\alpha_K} = \frac{\sigma(t_K)\dot{\varphi}(0)\sigma^2(t_K)}{\sigma(T_K)}(\chi_\gamma - \chi_{\gamma'}).$$

This yields the *combined formula of the two Euler-Savary equations*, namely

$$\left(\frac{1}{KP} - \frac{1}{KO_P}\right)\mathrm{sm}(g(K,P), t_K)\frac{\sigma^2(g(K,P))}{\sigma^2(t_K)\sigma(g(K,L))} = \dot{\varphi}(0)(\chi_\gamma - \chi_{\gamma'}) = \frac{\dot{\varphi}(0)}{\sigma^2(t_K)}\frac{1}{\alpha_K},$$

where we assume that $\sigma(T_K) = \mathrm{area}B = 1$.

dc_1387_17

CHAPTER 3

From the semi-indefinite inner product to the time-space manifold

The phrase "Minkowski space" do not distinguish between two theories: the theory of normed linear spaces and the theory of linear spaces with indefinite metric. For finite dimensions both are called Minkowski spaces in the literature. It is interesting that these essentially distinct theories of mathematics have similar axiomatic foundations. The axiomatic examination of the theory of linear spaces with indefinite metric comes from H. Minkowski [**123**] and the similar system of axioms of normed linear spaces was introduced by Lumer in [**108**]. The first concept widely used in physics: this is the mathematical structure of relativity theory and thus there is no doubt about its importance. The usability of the second one is based on the fact that modern functional analysis works in general normed spaces, and the Lumer-Giles theory of semi inner product gives a possibility to handling it by methods used originally in Hilbert spaces. We saw the basic properties of the latter in Section 2.2. The other concept is based on the following system of axioms. (See, e.g., [**66**].)

DEFINITION 3.0.16 ([**66**]). *The* indefinite inner product *(i.i.p.) on a complex vector space V is a complex function $[x, y] : V \times V \longrightarrow \mathbb{C}$ with the following properties:*

i1: *:* $[x + y, z] = [x, z] + [y, z]$,
i2: *:* $[\lambda x, y] = \lambda[x, y]$ *for every* $\lambda \in \mathbb{C}$,
i3: *:* $[x, y] = \overline{[y, x]}$ *for every* $x, y \in V$,
i4: *:* $[x, y] = 0$ *for every* $y \in V$ *then* $x = 0$.

A vector space V with an i.i.p. is called an i.i.p. space.

The standard mathematical model of space-time is a four dimensional i.i.p. space with signature $(+, +, +, -)$, also called Minkowski space in the literature. Thus we have a well known homonymism with the notion of Minkowski space!
Now we collect the common properties of the semi- and indefinite-inner-products and define the semi-indefinite inner product as well as the corresponding semi-indefinite inner product space. We also give a generalized concept of Minkowski space embedded in a semi-indefinite inner product space. In generalized Minkowski space and generalized space-time model with changing shape we investigate some important hypersurfaces giving a generalization either for n-dimensional hyperbolic space or the n-dimensional de Sitter space. Following our investigations in the Appendix we introduce the so-called time-space manifold, which is an analogous of the Lorentzian manifold in a generalized space-time model with changing shape and we give a version of general relativity theory valid in this structure.

3.1. Semi-indefinite inner product spaces

Let **s1**, **s2**, **s3**, **s4**, and **s5** be the five defining properties of an s.i.p. with the homogeneity property (see in Section 2.2). (As to the names: **s1** is the additivity property of the first argument, **s2** is the homogeneity property of the first argument, **s3** means the positivity of the function, **s4** is the Cauchy-Schwartz inequality and **s5** is the homogeneity property of the second argument.)
On the other hand, clearly **i1**=**s1**, **i2**=**s2**, and the properties **i3** and **i4** are the antisymmetry property and the nondegeneracy property of the indefinite inner product, respectively. It is easy to see that **s1**, **s2**, **s3**, **s5** imply **i4**, and if N is a positive (negative) subspace of an i.i.p. space, then **s4** holds on N. In the following definition we combine the concepts of s.i.p. and i.i.p..

DEFINITION 3.1.1 ([8]). *The* semi-indefinite inner product (s.i.i.p.) *on a complex vector space V is a complex function $[x,y]: V\times V\longrightarrow\mathbb{C}$ with the following properties:*

1: $[x+y,z]=[x,z]+[y,z]$ *(additivity in the first argument),*
2: $[\lambda x,y]=\lambda[x,y]$ *for every* $\lambda\in\mathbb{C}$ *(homogeneity in the first argument),*
3: $[x,\lambda y]=\overline{\lambda}[x,y]$ *for every* $\lambda\in\mathbb{C}$ *(homogeneity in the second argument),*
4: $[x,x]\in\mathbb{R}$ *for every* $x\in V$ *(the corresponding quadratic form is real-valued),*
5: *if either* $[x,y]=0$ *for every* $y\in V$ *or* $[y,x]=0$ *for all* $y\in V$*, then* $x=0$ *(nondegeneracy),*
6: $|[x,y]|^2\leq[x,x][y,y]$ *holds on non-positive and non-negative subspaces of V, respectively. (the Cauchy-Schwartz inequality is valid on positive and negative subspaces, respectively).*

A vector space V with a s.i.i.p. is called an s.i.i.p. space.

The interest in s.i.i.p. spaces depends largely on the example spaces given by the s.i.i.p. space structure.

EXAMPLE 3.1.1. We conclude that an s.i.i.p. space is a homogeneous s.i.p. space if and only if the property **s3** holds, too. An s.i.i.p. space is an i.i.p. space if and only if the s.i.i.p. is an antisymmetric product. In this latter case $[x,x]=\overline{[x,x]}$ implies **4**, and the function is also Hermitian linear in its second argument. In fact, we have: $[x,\lambda y+\mu z]=\overline{[\lambda y+\mu z,x]}=\overline{\lambda}\overline{[y,x]}+\overline{\mu}\overline{[z,x]}=\overline{\lambda}[x,y]+\overline{\mu}[x,z]$. It is clear that both of the classical "Minkowski spaces" can be represented either by an s.i.p or by an i.i.p., so automatically they can also be represented as an s.i.i.p. space.

EXAMPLE 3.1.2. In an arbitrary complex normed linear space V we can define an s.i.i.p. which is a generalization of a representing s.i.p. of the norm function. Let now C be the unit sphere of the space V. By the Hahn-Banach theorem there exists at least one continuous linear functional, and we choose exactly one such that $\|\widetilde{v}^\star\|=1$ and $\widetilde{v}^\star(v)=1$. Consider a sign function $\varepsilon([v])$ with value ±1 on $C/\sim$, where $C/\sim$ means the factorization of C by the equivalence relation

$$"x\sim y\Leftrightarrow x=\lambda y \text{ with a nonzero } \lambda".$$

If now $\varepsilon([v])=1$ let it be denoted by $v^\star=\widetilde{v}^\star$, and $\varepsilon([v])=-1$ defines $v^\star=-\widetilde{v}^\star$. Finally, extend it homogeneously to V by the equality $(\lambda v)^\star=\overline{\lambda}v^\star$. Of course, for an arbitrary vector v of V the corresponding linear functional satisfies the equalities $v^\star(v):=\varepsilon([v])\|v\|^2$ and $\|v\|=\|v^\star\|$. Now the function

$$[u,v]=v^\star(u)$$

satisfies **1-5**. If U is a non-negative subspace, then it is positive and we have for all nonzero $u,v\in U$ that

$$|[u,v]|=|v^\star(u)|=\frac{|v^\star(u)|}{\|u\|}\|u\|\leq\|v^\star\|\|u\|=\|v\|\|u\|,$$

proving **6**.
To define the generalized Minkowski space we need a lemma:

LEMMA 3.1.1 ([8]). *Let $(S,[\cdot,\cdot]_S)$ and $(T,-[\cdot,\cdot]_T)$ be two s.i.p. spaces. Then the function $[\cdot,\cdot]^-:(S+T)\times(S+T)\longrightarrow\mathbb{C}$ defined by*

$$[s_1+t_1,s_2+t_2]^-:=[s_1,s_2]-[t_1,t_2]$$

is an s.i.p. on the vector space $S+T$.

PROOF. The function $[\cdot,\cdot]^-$ is non-negative, as we can easily see from its definition. First we prove the linearity in the first argument. We have

$$[\lambda'(s'+t')+\lambda''(s''+t''),s+t]^-=[\lambda's'+\lambda''s'',s]_S-[\lambda't'+\lambda''t'',t]_T=$$
$$=\lambda'[s',s]_S+\lambda''[s'',s]_S-\lambda'[t',t]_T-\lambda''[t'',t]_T=\lambda'[s'+t',s+t]^-+\lambda''[s''+t'',s+t]^-.$$

The homogeneity in the second argument is trivial. In fact, we have

$$[s'+t',\lambda(s+t)]^- = [s',\lambda s]_S - [t',\lambda t]_T = \overline{\lambda}[s'+t',s+t]^-.$$

Finally we check the Cauchy-Schwartz inequality. We have

$$\begin{aligned}|[s_1+t_1,s_2+t_2]^-|^2 = [s_1+t_1,s_2+t_2]^-\overline{[s_1+t_1,s_2+t_2]^-} = ([s_1,s_2]_S-[t_1,t_2]_T)(\overline{[s_1,s_2]_S}-\overline{[t_1,t_2]_T}) = \\ = [s_1,s_2]_S\overline{[s_1,s_2]_S} + [t_1,t_2]_T\overline{[t_1,t_2]_T} + [s_1,s_2]_S(-\overline{[t_1,t_2]_T}) + (-[t_1,t_2]_T)\overline{[s_1,s_2]_S} \leq \\ \leq [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\mathrm{Re}\{[s_1,s_2]_S(-\overline{[t_1,t_2]_T})\} \leq \\ \leq [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2|[s_1,s_2]_S||[t_1,t_2]_T| \leq \\ \leq [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\sqrt{[s_1,s_1]_S[s_2,s_2]_S[t_1,t_1]_T[t_2,t_2]_T},\end{aligned}$$

and by the inequality between the arithmetic and geometric means we get that

$$\begin{aligned}[s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + 2\sqrt{[s_1,s_1]_S[s_2,s_2]_S[t_1,t_1]_T[t_2,t_2]_T} \leq \\ \leq [s_1,s_1]_S[s_2,s_2]_S + [t_1,t_1]_T[t_2,t_2]_T + [s_1,s_1]_S(-[t_2,t_2]_T + (-[t_1,t_1]_T)[s_2,s_2]_S = \\ = ([s_1,s_1]_S - [t_1,t_1]_T)([s_2,s_2]_S - [t_2,t_2]_T) = [s_1+t_1,s_1+t_1]^-[s_2+t_2,s_2+t_2]^-.\end{aligned}$$

□

It is possible that the s.i.i.p. space V is a direct sum of its two subspaces where one of them is positive and the other one is negative. Then we have two more structures on V, an s.i.p. structure (by Lemma 3.1.1) and a natural third one, which we will call Minkowskian structure. More precisely, we have

DEFINITION 3.1.2 ([8]). *Let $(V,[\cdot,\cdot])$ be an s.i.i.p. space. Let $S,T \leq V$ be positive and negative subspaces, where T is a direct complement of S with respect to V. Define a product on V by the equality $[u,v]^+ = [s_1+t_1,s_2+t_2]^+ = [s_1,s_2]+[t_1,t_2]$, where $s_i \in S$ and $t_i \in T$, respectively. Then we say that the pair $(V,[\cdot,\cdot]^+)$ is a* generalized Minkowski space *with Minkowski product $[\cdot,\cdot]^+$. We also say that V is a real generalized Minkowski space if it is a real vector space and the s.i.i.p. is a real valued function.*

The Minkowski product defined by the above equality satisfies properties **1-5** of the s.i.i.p.. But in general, property **6** does not hold. To see this, define an s.i.i.p. space in the following way:

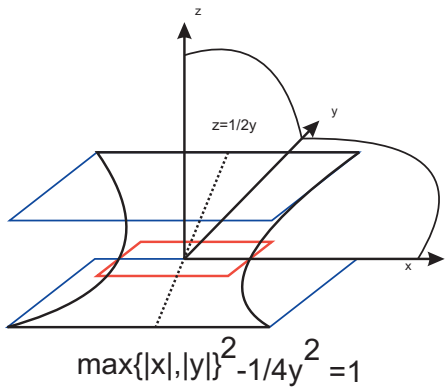

FIGURE 3.1. The unit sphere of a positive subspace of the Example

Consider a 2-dimensional L^∞ space S of the embedding three dimensional Euclidean space E^3. Choose an orthonormed basis $\{e_1,e_2,e_3\}$ of E^3 for which $e_1,e_2 \in S$, and give an s.i.p. associated to the L^∞ norm as follows:

$$[x_1e_1+x_2e_2,y_1e_1+y_2e_2]_S := x_1y_1\lim_{p\to\infty}\frac{1}{\left(1+\left(\frac{y_2}{y_1}\right)^p\right)^{\frac{p-2}{p}}} + x_2y_2\lim_{p\to\infty}\frac{1}{\left(1+\left(\frac{y_1}{y_2}\right)^p\right)^{\frac{p-2}{p}}}.$$

By Lemma 3.1.1 the function

$$[x_1e_1+x_2e_2+x_3e_3,y_1e_1+y_2e_2+y_3e_3]^- := [x_1e_1+x_2e_2,y_1e_1+y_2e_2]_S + x_3y_3$$

is an s.i.p. on E^3 associated to the norm

$$\sqrt{[x_1e_1+x_2e_2+x_3e_3, x_1e_1+x_2e_2+x_3e_3]^-} := \sqrt{\max\{|x_1|,|x_2|\}^2+x_3^2}.$$

By the method of Example 3.1.2 consider such a sign function for which $\varepsilon(v)$ is equal to 1 if v is in $S\cap C$, and is equal to -1 if $v=e_3$ holds. (C denotes the unit sphere, as in the previous examples.) This sign function determines an s.i.i.p. $[\cdot,\cdot]$ and thus generates a Minkowski product $[\cdot,\cdot]^+$, for which the corresponding square root function is

$$f(v) := \sqrt{[x_1e_1+x_2e_2+x_3e_3, x_1e_1+x_2e_2+x_3e_3]^+} = \sqrt{\max\{|x_1|,|x_2|\}^2-x_3^2}.$$

As it can be easily seen, the plane $x_3=\alpha x_2$ for $0<\alpha<1$ is a positive subspace with respect to the Minkowski product, but its unit ball is not convex (see Fig. 3.1).
But $f(v)$ is homogeneous, correspondingly it is not subadditive. Since the Cauchy-Scwartz inequality implies subadditivity, this inequality remains false in this positive subspace.

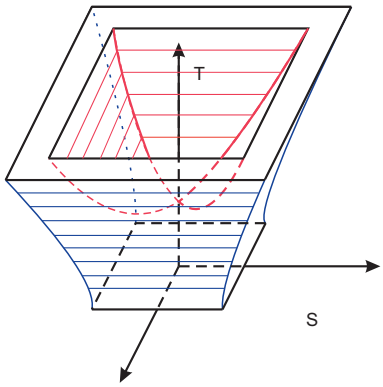

FIGURE 3.2. The case of the norm L_∞.

Giles in [**64**] gave an associated s.i.p. for L_p spaces. Using the method of our Example 3.1.2, we can define s.i.i.p. spaces based on the L_p structure. Let $(S,[\cdot,\cdot]_S)$ be the s.i.p. space, where S is the real Banach space $L_{p_1}(X,\mathcal{S},\mu)$ and T is the real Banach space $L_{p_2}(Y,\mathcal{S}',\nu)$, respectively. If $1<p_1,p_2\le\infty$, then these spaces can be readily expressed, as a uniform s.i.p. space with s.i.p. defined by

$$[s_1,s_2]_S = \frac{1}{\|s_2\|_{p_1}^{p_1-2}}\int_X s_1|s_2|^{p_1-1}\mathrm{sgn}\ (s_2)d\mu$$

and

$$[t_1,t_2]_T = \frac{1}{\|t_2\|_{p_2}^{p_2-2}}\int_Y t_1|t_2|^{p_2-1}\mathrm{sgn}\ (t_2)d\nu,$$

respectively. Consider the real vector space $S+T$ with the s.i.p.

$$[u,v]^- := [s_1,s_2]_S+[t_1,t_2]_T.$$

This is also a uniform s.i.p. space, since in Lemma 3.1.1 we proved that it is an s.i.p. space and

$$|[z,x]-[z,y]| = |([s_3,s_1]_S-[s_3,s_2]_S)+([t_3,t_1]_T-[t_3,t_2]_T)| \le$$
$$\le |[s_3,s_1]_S-[s_2,s_1]_S|+|[t_3,t_1]_T-[t_2,t_1]_T| \le 2(p_1-1)\|s_1-s_2\|_{p_1}+2(p_2-2)\|t_1-t_2\|_{p_2},$$

implying that the space is uniformly continuous. It has been established that such spaces are uniformly convex (see [**38**], p. 403). We could define an s.i.i.p space on $S+T$ such that the subspace S is positive and T is a negative one, and a Minkowski space by the Minkowski product $[u,v]^+ := [s_1,s_2]_S-[t_1,t_2]_T$, respectively. (In Fig. 3.2 one can see the case when $\dim S=\dim T+1=2$ and the norm of S is L_∞.)
We define the orthogonality of such a space by a definition analogous to the definition of the orthogonality of an i.i.p. or s.i.p. space.

DEFINITION 3.1.3 ([8]). *The vector v is orthogonal to the vector u if $[v,u]=0$. If U is a subspace of V, define the orthogonal companion of U in V by $U^{\perp}=\{v\in V|[v,u]=0$ for all $u\in U\}$. A vector v is* neutral vector *if $[v,v]=0$.*

We note that, as in the i.i.p. case, the orthogonal companion is always a subspace of V.

THEOREM 3.1.1. [8] *Let V be an n-dimensional s.i.i.p. space. Then the orthogonal companion of a non-neutral vector u is a subspace having a direct complement of the linear hull of u in V. The orthogonal companion of a neutral vector v is a degenerate subspace of dimension $n-1$ containing v.*

We omit the easy proof.

REMARK 3.1.1. *The proof of Theorem 3.1.1 does not use the property* **6** *of the s.i.i.p.. So this statement is true for any concepts of product satisfying properties* **1**-**5**. *As we saw, the Minkowski product is also such a product. It can be proved also that in a generalized Minkowski space, the positive and negative components S and T are Pythagorean orthogonal to each other. In fact, for every pair of vectors $s\in S$ and $t\in T$, by definition we have $[s-t,s-t]^{+}=[s,s]+[-t,-t]=[s,s]^{+}+[t,t]^{+}$.*

The following theorem will be a common generalization of the theorem on diameters conjugated to each other in a real, finite dimensional normed linear space, and a theorem on the existence of an orthogonal system in an i.i.p. space. A set of n diameters of the unit ball of an n-dimensional real normed space is considered to be a set of conjugate diameters if their normalized vectors have the following property: Choosing one of them, each vector in the linear span of the remaining direction vectors is orthogonal to it. An *Auerbach basis* of a normed space is a set of direction vectors having this property. Any real normed linear space has at least two Auerbach bases. One is induced by a cross-polytope inscribed in the unit ball of maximal volume (see [139]), and the other one by the midpoints of the facets of a circumscribed parallelotope of minimum volume (see [40]). These two ways of finding Auerbach bases are dual in the sense that if an Auerbach basis is induced by an inscribed cross-polytope of maximum volume, then any dual basis is induced by a circumscribed parallelotope of minimum volume, and vice versa (cf. [95]). If any minimum volume basis and maximum volume basis coincide, then by a result of Lenz (see [102]) we have that the space is a real i.p. space of finite dimension.
For generalized Minkowski spaces we have an analogous theorem which straightforward proof we omit here.

THEOREM 3.1.2. [8] *In a finite dimensional, real, generalized Minkowski space there is a basis with the Auerbach property. In other words, its vectors are orthogonal to the $(n-1)$-dimensional subspace spanned by the remaining ones. For this basis there is a natural number k, less or equal to n, for which $\{e_1,\dots,e_k\}\subset S$ and $\{e_{k+1},\dots,e_n\}\subset T$. Finally, this basis has also the Auerbach property in the s.i.p. space $(V,[\cdot,\cdot]^{-})$.*

3.2. Generalized space-time model

It is easy to see that by this method, starting with arbitrary two normed spaces S and T, one can mix a generalized Minkowski space. Of course its smoothness property is basically determined by the analogous properties of S and T.

DEFINITION 3.2.1 ([8]). *Let V be a generalized Minkowski space. Then we call a vector space-like, light-like, or time-like if its scalar square is positive, zero, or negative, respectively. Let $\mathcal{S},\mathcal{L}$ and $\mathcal{T}$ denote the sets of the space-like, light-like, and time-like vectors, respectively. In a finite dimensional, real generalized Minkowski space with $\dim T=1$ is called* generalized space-time model.

In the case of generalized space-time model we can geometrically characterize these sets of vectors. At this time $\mathcal{T}$ is a union of its two parts, namely $\mathcal{T}=\mathcal{T}^{+}\cup\mathcal{T}^{-}$, where

$$\mathcal{T}^{+}=\{s+t\in\mathcal{T}|\text{ where } t=\lambda e_n \text{ for } \lambda\geq 0\} \text{ and } \mathcal{T}^{-}=\{s+t\in\mathcal{T}|\text{ where } t=\lambda e_n \text{ for } \lambda\leq 0\}.$$

THEOREM 3.2.1 ([8]). *Let V be a generalized space-time model. Then $\mathcal{T}$ is an open double cone with boundary $\mathcal{L}$, and the positive part $\mathcal{T}^+$ (resp. negative part $\mathcal{T}^-$) of $\mathcal{T}$ is convex.*

PROOF. The conic property immediately follows from the equality $[\lambda v, \lambda v]^+ = \lambda\overline{\lambda}[v,v]^+ = |\lambda|^2[v,v]^+$. Consider now the affine subspace of dimension $(n-1)$ which is of the form $U = S+t$, where $t \in T$ is arbitrary, but non zero. Then, for an element of $\mathcal{T} \bigcap U$, we have $0 \geq [s+t, s+t]^+ = [s,s] + [t,t]$, and therefore $[s,s] \leq -[t,t]$. This implies that the above intersection is a convex body in the $(n-1)$-dimensional real vector space S. The s.i.i.p. in S induces a norm whose unit ball is a centrally symmetric convex body. So $\mathcal{T}$ is a double cone and its positive (resp. negative) part is convex, as we stated. For the vectors of its boundary equality holds, and so these are light-like vectors. Since those vectors of the space, for which the inequality does not hold, are space-time vectors, we also get the remaining statement of the theorem. □

In the rest of the paper [8] we considered a special subset, the imaginary unit sphere of a finite dimensional, real, generalized Minkowski space. (Some steps of our investigation are also valid in a complex generalized Minkowski space. If we do not use the attribute "real", then we think about a complex Minkowski space.) We give a metric on it, and thus we will get a structure similar to the hyperboloid model of the hyperbolic space embedded in a space-time model.
We note that if $\dim T > 1$ or the space is complex, then the set of time-like vectors cannot be divided into two convex components. So we have to consider that our space is a generalized space-time model.

3.2.1. The imaginary unit sphere. It is known that in a Lorentzian space the imaginary unit sphere can be identified with the $n-1$-dimensional hyperbolic space. Hence the imaginary unit sphere of a generalized space-time model can be considered as a generalization of the hyperbolic space. We begin with a definition:

DEFINITION 3.2.2 ([9]). *The set*

$$H := \{v \in V | [v,v]^+ = -1\},$$

is called the imaginary unit sphere.

With respect to the embedding real normed linear space $(V, [\cdot,\cdot]^-)$ (see Lemma 3.1.1) H is, as we saw, a generalized two sheets hyperboloid corresponding to the two pieces of $\mathcal{T}$, respectively. Usually we deal only with one sheet of the hyperboloid, or identify the two sheets projectively. In this case the space-time component $s \in S$ of v determines uniquely the time-like one, namely $t \in T$. Let $v \in H$ be arbitrary. Let T_v denote the set $v + v^\perp$, where $v^\perp$ is the orthogonal complement of v with respect to the s.i.i.p., thus a subspace.

THEOREM 3.2.2 ([8]). *The set T_v corresponding to the point $v = s + t \in H$ is a positive, $(n-1)$-dimensional affine subspace of the generalized space-time model $(V, [\cdot,\cdot]^+)$.*

PROOF. By the definition of H the component t of v is non-zero. As we saw in Theorem 3.1.1, if $[v,v] \neq 0$, then $v^\perp$ is an $(n-1)$-dimensional subspace of V. Let now $w \in T_v - v$ be an arbitrary vector. We have to prove that if $[v,v] = -1$ and w is orthogonal to v, then $[w,w] > 0$. Let now $w = s' + t'$ and assume that $[t',t'] = 0$. Then, by the definition of T, $t' = 0$ and thus $[w,w] = [s,s] > 0$ holds. In this case, we may assume that $[t',t'] \neq 0$, and so $t' = \lambda t$. On the other hand, we have $0 = [w,v]^+ = [s',s] + [t',t]$. We can use the Cauchy-Schwartz inequality for the space-time components, and we have

$$[s,s][s',s'] \geq |[s',s]|^2 = |-[t',t]|^2 = |\lambda|^2|-[t,t]|^2 = |\lambda|^2[t,t]^2.$$

Since $[s,s][t',t'] = \lambda\overline{\lambda}[s,s][t,t] = |\lambda|^2[s,s][t,t]$, we get the inequality

$$[s,s][w,w]^+ = [s,s]([s',s'] + [t',t']) \geq |\lambda|^2([t,t]^2 + [s,s][t,t]).$$

By the definition of H we also have $-1 = [v,v]^+ = [s,s] + [t,t]$ and

$$[s,s][w,w]^+ \geq |\lambda|^2([t,t]^2 + (-1-[t,t])[t,t]) = -|\lambda|^2[t,t] > 0.$$

Consequently, if s is nonzero then $[w,w]>0$, as we stated. If now $[s,s]=0$ then $[t,t]=-1$, and $0=[s'+t',t]=[s',t]+[t',t]=[t',t]$ implies that $t'=0$ and $w\in S$. Thus we proved the statement. □

Each of the affine spaces T_v of H can be considered as a semi-metric space, where the semi-metric arises from the Minkowski product restricted to this positive subspace of V. We recall that the Minkowski product does not satisfy the Cauchy-Schwartz inequality. Thus the corresponding distance function does not satisfy the triangle inequality. Such a distance function is called in the literature semi-metric (see [138]). Thus, if the set H is sufficiently smooth, then a metric can be adopted for it, which arises from the restriction of the Minkowski product to the tangent spaces of H. Let us discuss this more precisely.
The directional derivatives of a function $f:S\longmapsto\mathbb{R}$ with respect to a unit vector e of S can be defined in the usual way, by the existence of the limits for real λ: $f'_e(s)=\lim\limits_{\lambda\mapsto 0}\frac{f(s+\lambda e)-f(s)}{\lambda}$.
Let now the generalized Minkowski space be a generalized space-time model, and consider a mapping f on S to $\mathbb{R}$ and a basis $\{e_1,\dots,e_n\}$. The set of points $F:=\{(s+f(s)e_n)\in V \text{ for } s\in S\}$ is a so-called *hypersurface* of this space. Tangent vectors of a hypersurface F in a point p are the vectors associated to the directional derivatives of the coordinate functions in the usual way. So u is a *tangent vector* of the hypersurface F in its point $v=(s+f(s)e_n)$, if it is of the form $u=\alpha(e+f'_e(s)e_n)$ for real α and unit vector $e\in S$. The linear hull of the tangent vectors translated into the point s is the tangent space of F in s. If the tangent space has dimension $(n-1)$ we call it *tangent hyperplane*. It can be seen easily, that the explicit form of H arises from the function

$$f:s\longmapsto\sqrt{1+[s,s]}.$$

Since its directional derivatives can be concretely determined, we can give a connection between the differentiability properties and the orthogonality one.

LEMMA 3.2.1 ([8]). *Let V be a generalized Minkowski space and assume that the s.i.p. $[\cdot,\cdot]|_S$ is continuous. (So the property* **s6** *holds.) Then the directional derivatives of the real valued function $f:s\longmapsto\sqrt{1+[s,s]}$ are $f'_e(s)=\frac{\mathrm{Re}[e,s]}{\sqrt{1+[s,s]}}$.*

PROOF.
The considered derivative is

$$\frac{f(s+\lambda e)-f(s)}{\lambda}=\frac{\sqrt{1+[s+\lambda e,s+\lambda e]}-\sqrt{1+[s,s]}}{\lambda}=\frac{\sqrt{1+[s+\lambda e,s+\lambda e]}\sqrt{1+[s,s]}-(1+[s,s])}{\lambda\sqrt{1+[s,s]}}.$$

Since $s+\lambda e,s\in S$, and S is a positive subspace, we have

$$0\le(\sqrt{[s+\lambda e,s+\lambda e]}-\sqrt{[s,s]})^2=[s+\lambda e,s+\lambda e]-2\sqrt{[s+\lambda e,s+\lambda e]}\sqrt{[s,s]}+[s,s],$$

and so $[s+\lambda e,s+\lambda e]+[s,s]\ge 2\sqrt{[s+\lambda e,s+\lambda e]}\sqrt{[s,s]}\ge 2|[s+\lambda e,s]|$, yielding also $[s+\lambda e,s+\lambda e]+[s,s]\ge 2|[s,s+\lambda e]|$. Using these inequalities, we get that

$$\frac{f(s+\lambda e)-f(s)}{\lambda}\ge\frac{\sqrt{1+2|[s+\lambda e,s]|+|[s+\lambda e,s]|^2}-(1+[s,s])}{\lambda\sqrt{1+[s,s]}}=$$

$$\frac{1+|[s+\lambda e,s]|-1-[s,s]}{\lambda\sqrt{1+[s,s]}}\ge\frac{\mathrm{Re}\{[s,s]+\lambda[e,s]\}-[s,s]}{\lambda\sqrt{1+[s,s]}}=\frac{\mathrm{Re}[e,s]}{\sqrt{1+[s,s]}}.$$

But also

$$\frac{f(s+\lambda e)-f(s)}{\lambda}=\frac{(1+[s+\lambda e,s+\lambda e])-\sqrt{1+[s,s]}\sqrt{(1+[s+\lambda e,s+\lambda e])}}{\lambda\sqrt{1+[s+\lambda e,s+\lambda e]}}\le$$

$$\le\frac{(1+[s+\lambda e,s+\lambda e])-1-|[s,s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e,s+\lambda e]}}=\frac{\mathrm{Re}\{[s+\lambda e,s+\lambda e]\}-|[s,s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e,s+\lambda e]}}=$$

$$=\frac{\mathrm{Re}\{[s,s+\lambda e]+\lambda[e,s+\lambda e]\}-|[s,s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e,s+\lambda e]}}\le$$

$$\leq \frac{|[s,s+\lambda e]|+\mathrm{Re}\{\lambda[e,s+\lambda e]\}-|[s,s+\lambda e]|}{\lambda\sqrt{1+[s+\lambda e,s+\lambda e]}}=\frac{\mathrm{Re}\{[e,s+\lambda e]\}}{\sqrt{1+[s+\lambda e,s+\lambda e]}}.$$

Now the continuity property **s6** implies that the examined limit exists, and that the differential is $\frac{\mathrm{Re}[e,s]}{\sqrt{1+[s,s]}}$, as we stated. □

The following theorem is a consequence of this result.

THEOREM 3.2.3 ([9]). *Let assume that the s.i.p. $[\cdot,\cdot]$ of S is differentiable. (So the property* **s6'** *holds.) Then for every two vectors x and z in S we have:*

$$[x,\cdot]'_z(x)=2\mathrm{Re}[z,x]-[z,x],\ \textit{and}\ \|\cdot\|''_{x,z}(x)=\frac{\mathrm{Re}[z,x]-[z,x]}{\|x\|}.$$

If we also assume that the s.i.p. is continuously differentiable (so the norm is a C^2 function), then we also have

$$[x,\cdot]'_x(y)=[x,x]\ \textit{and thus}\ \|\cdot\|''_{x,x}(y)=\|x\|^2-\frac{\mathrm{Re}[x,y]^2}{\|y\|^2}.$$

PROOF. Since

$$\frac{1}{\lambda}\left([x+\lambda z,x+\lambda z]-[x,x]\right)=\frac{1}{\lambda}\left([x,x+\lambda z]-[x,x]\right)+\frac{1}{\lambda}[\lambda z,x+\lambda z],$$

if λ tends to zero then the right hand side tends to $[x,\cdot]'_z(x)+[z,x]$. The left hand side is equal to

$$\frac{\left(\sqrt{1+[x+\lambda z,x+\lambda z]}-\sqrt{1+[x,x]}\right)\left(\sqrt{1+[x+\lambda z,x+\lambda z]}+\sqrt{1+[x,x]}\right)}{\lambda}$$

thus by Lemma 3.2.1 it tends to

$$\frac{\mathrm{Re}[z,x]}{\sqrt{1+[x,x]}}2\sqrt{1+[x,x]}.$$

This implies the first equality $[x,\cdot]'_z(x)=2\mathrm{Re}[z,x]-[z,x]$. Using Theorem 2.2.1 we also get that $\|x\|(\|\cdot\|''_{x,z}(x))=[x,\cdot]'_z(x)-\frac{\mathrm{Re}[x,x]\mathrm{Re}[z,x]}{\|x\|^2}$, proving the second statement, too.
If we assume that the norm is a C^2 function of its argument then the first derivative of the second argument of the product is a continuous function of its arguments. So the function $A(y):S\longrightarrow\mathbb{R}$ defined by the formula

$$A(y)=[x,\cdot]'_x(y)=\lim_{\lambda\mapsto 0}\frac{1}{\lambda}\left([x,y+\lambda x]-[x,y]\right)$$

continuous in $y=0$. On the other hand for non-zero $t\in\mathbb{R}$ we use the notation $t\lambda'=\lambda$ and we get that

$$A(ty)=\lim_{\lambda\mapsto 0}\frac{1}{\lambda}\left([x,ty+\lambda x]-[x,y]\right)=\lim_{\lambda'\mapsto 0}\frac{t}{t\lambda'}\left([x,y+\lambda' x]-[x,y]\right)=A(y).$$

From this we can see immediately that $[x,\cdot]'_x(y)=A(y)=A(0)=[x,x]$ holds for every y. Applying again the formula connected the derivative of the product and the norm we get the last statement of the theorem, too. □

Now we apply our investigation in a generalized space-time model to H. We can give a connection between the differentiability properties and the orthogonality one.

LEMMA 3.2.2 ([8]). *Let H be the imaginary unit sphere of a generalized space-time model. Then the tangent vectors of the hypersurface H in its point $v=s+\sqrt{1+[s,s]}e_n$ form the orthogonal complement $v^\perp$ of v.*

PROOF. A tangent vector of this space is of the form $u = \alpha(e + f'_e(s)e_n)$, where by the previous lemma $f'_e(s) = \frac{\mathrm{Re}[e,s]}{\sqrt{1+[s,s]}} = \frac{[e,s]}{\sqrt{1+[s,s]}}$. Thus we have

$$\left[\alpha\left(e + \frac{[e,s]}{\sqrt{1+[s,s]}}e_n\right), s+t\right]^+ = \alpha[e,s] + \alpha\left[\frac{[e,s]}{\sqrt{1+[s,s]}}e_n, \sqrt{1+[s,s]}e_n\right] = \alpha([e,s] - [e,s]) = 0.$$

So the tangent vectors are orthogonal to the vector v. Conversely, if for a vector $u = s' + t' = s' + \lambda e_n$ we have $0 = [u,v] = [s',s] + [t',t]$ then $[s',s] = -[\lambda e_n, t] = \lambda\sqrt{1+[s,s]}$, since $-[t,t] = 1 + [s,s]$ by the definition of H. Introducing the notion $e = \frac{s'}{\sqrt{[s',s']}}$, we get that

$$[e,s] = \left[\frac{s'}{\sqrt{[s',s']}}, s\right] = \frac{\lambda}{\sqrt{[s',s']}}\sqrt{1+[s,s]},$$

implying that

$$\frac{\lambda}{\sqrt{[s',s']}} = \frac{[e,s]}{\sqrt{1+[s,s]}} = f'_e(s).$$

In this way

$$u = \sqrt{[s',s']}\left(\frac{s'}{\sqrt{[s',s']}} + \frac{\lambda}{\sqrt{[s',s']}}e_n\right) = \alpha(e + f'_e(s)e_n).$$

This last equality shows that a vector of the orthogonal complement is a tangent vector, as we stated. □

We defined the Finsler space type structure for a hypersurface of a generalized space-time model.

DEFINITION 3.2.3 ([8]). *Let F be a hypersurface of a generalized space-time model for which the following properties hold:*

- **i:** *In every point v of F, there is a (unique) tangent hyperplane T_v for which the restriction of the Minkowski product $[\cdot,\cdot]_v^+$ is positive, and*
- **ii:** *the function $ds_v^2 := [\cdot,\cdot]_v^+ : F \times T_v \times T_v \longrightarrow \mathbb{R}^+$ $ds_v^2 : (v, u_1, u_2) \longmapsto [u_1,u_2]_v^+$ varies differentiable with the vectors $v \in F$ and $u_1, u_2 \in T_v$.*

Then we say that the pair (F, ds^2) is a Minkowski-Finsler space with semi-metric ds^2 embedding into the generalized space-time model V.

Naturally "varies differentiable with the vectors v, u_1, u_2" means that for every $v \in T$ and pairs of vectors $u_1, u_2 \in T_v$ the function $[u_1,u_2]_v$ is a differentiable function on F.
Assume now that the s.i.i.p. restricted into S is continuously differentiable. In a connected Finsler space any point has a distance from any other point of the space (see e.g. [138]). By our terminology the distance can be computed in the following analogous way.

DEFINITION 3.2.4 ([8]). *Denote by p, q a pair of points in H^+ and consider the set $\Gamma_{p,q}$ of equally oriented piecewise differentiable curves $c(t)$ $a \le t \le b$ of H^+ emanating from p and terminating at q. Then the Minkowskian-Finsler distance of these points is*

$$\rho(p,q) = \inf\left\{\int_a^b \sqrt{[\dot{c}(x), \dot{c}(x)]^+_{c(x)}}dx \text{ for } c \in \Gamma_{p,q}\right\},$$

where $\dot{c}(x)$ means the tangent vector of the curve c at its point $c(x)$.

We would like to examine the influence of a linear isometry to the Minkowski-Finsler distance. It is easy to see that this distance satisfies the triangle inequality; thus it is a metric on H^+ (see [138]).

DEFINITION 3.2.5 ([8]). *A topological isometry $f : H \longrightarrow H$ of H is a homeomorphism of H which preserves the Minkowski-Finsler distance between each pair of points of H.*

We note that in this definition a linear mapping F restricted to S gives an isometry between S and its image $F(S)$ implying that this image is a normed space with respect to those s.i.p. which raised from the s.i.p. of S. This isometry is stronger than the usual one, in which we need only the equality of the norm of the corresponding vectors. As we can see earlier (Theorem 2.2.5) Koehler theorem says that a mapping in a smooth Banach space is an isometry if and only if it preserves the (unique) s.i.p.. Thus, if the norm is at least smooth, then the two types of linear isometry coincide. Koehler also proved [**97**] that if the generalized Riesz-Fischer representation theorem is valid in a normed space, then every bounded linear operator A has a generalized adjoint A^T defined by the equality $[A(x), y] = [x, A^T(y)]$ for all $x, y \in V$. This mapping is the usual Hilbert space adjoint if the space is an i.p. space. In this more general setting this map is not usually linear but it still has some interesting properties. The assumption for the s.i.p. in Koehler paper [**97**] is that the space should be a smooth and uniformly convex Banach space. It is well known that uniform convexity implies strict convexity. On the other hand, we now take also into consideration (see [**144**] p. 111) that every, strictly convex, finite-dimensional normed vector space is uniformly convex. So for the rest of the section we shall assume that the normed space S with respect to its s.i.p. is strictly convex and smooth. It is convenient to characterize strict convexity of the norm in terms of s.i.p. properties. E. Berkson [**24**] states, what can be simply proved, namely

LEMMA 3.2.3 ([**24**]). *An s.i.p. space is strictly convex if and only if* $[x, y] = \|x\|\|y\|$ *with* $x, y \neq 0$ *implies* $y = \lambda x$ *for some real* $\lambda > 0$.

The following theorem is true for the imaginary unit sphere.

THEOREM 3.2.4 ([**8**]). *Let V be a generalized space-time model.*

- *If S is a continuously differentiable s.i.p. space, then (H^+, ds^2) is a Minkowski-Finsler space.*
- *If we assume that the subspace S is a strictly convex, smooth normed space with respect to the norm associated to the s.i.i.p. then the s.i.p. space $\{V, [\cdot,\cdot]^-\}$ is also smooth and strictly convex. Let F^T be the generalized adjoint of the linear mapping F with respect to the s.i.p. space $\{V, [\cdot,\cdot]^-\}$, and define the involutory linear mapping $J : V \longrightarrow V$ by the equalities $J|S = id|S$, $J|T = -id|_T$. The map $F|_H = f : H \longrightarrow H$ is a linear isometry of the upper sheet H^+ of H if and only if it is invertible, satisfies the equality: $F^{-1} = JF^TJ$, and, moreover, takes e_n into a point of H^+.*
- *A linear isometry of H^+ is also a topological isometry on it.*
- *Assume that also that the group of linear isometries of H^+ acts transitively on H^+. Denote the Minkowski-Finsler distance of H^+ by $d(\cdot,\cdot)$. Then the following statement is true: $[a, b]^+ = -ch(d(a, b))$ for $a, b \in H^+$.*

PROOF. If the s.i.p. of S is a continuously differentiable one, then the norm is twice differentiable (see Theorem 2.2.1). This also implies the continuity of the s.i.p., and so we know by Lemma 3.2.1 that there is a unique tangent hyperplane at each point of H. By Theorem 3.2.2 we get that the Minkowski product restricted to a tangent hyperplane is positive. So the first assumption of the definition is valid.
To prove the second condition, consider the product $[u_1, u_2]_v^+$, where v is a point of H and u_1,u_2 are two vectors on its tangent hyperplane. Then, by Lemma 3.2.1, we have:

$$u_i = \alpha_i \left(s_i + \frac{[s_i, s_v]}{\sqrt{1 + [s_v, s_v]}} e_n \right) \text{ for } i = 1, 2.$$

Here the vectors s_1, s_2, s_v are in S and $v = s_v + \sqrt{1 + [s_v, s_v]}e_n$. Thus the examined product is

$$[u_1, u_2]_v^+ = \alpha_1\alpha_2 \frac{[s_1, s_2](1 + [s_v, s_v]) - [s_1, s_v][s_2, s_v]}{(1 + [s_v, s_v])}.$$

Since the function $[s_v, s_v] = ([v, e_n]^+)^2 - 1$ is a continuously differentiable function of v, and $[s_1, s_2]$ is (by our assumption) also a continuously differentiable function of its arguments, we have to prove, that the map sending u_i to s_i also has this property. But this latter fact is a consequence of the observation that the map $u \mapsto s$ is a projection, and so it is linear.

To prove the statements of the second item firstly we notes that the embedding normed space $\{V, [\cdot,\cdot]^-\}$ is also smooth and strictly convex. The equality $1 = [s+t, s+t]^- = [s,s] - [t,t] = [s,s] + \|t\|^2$ shows that the unit balls of the two norms are smooth at the same time. To prove strict convexity, consider $[s+t, s'+t']^- = \|s+t\|^-\|s'+t'\|^-$. Since $\dim T = 1$, we can assume that $t' = \lambda t$ for some real λ. Thus we get the equality

$$[s,s][s',s'] = [s,s']^2 + [t,t]([s',s'] - 2\lambda[s,s'] + \lambda^2[s,s]).$$

By the Cauchy-Schwartz inequality we have

$$[s',s'] - 2\lambda[s,s'] + \lambda^2[s,s] \geq \left(\sqrt{[\lambda s, \lambda s]} - \sqrt{[s',s']}\right)^2 \geq 0,$$

and so

$$0 \leq [s,s']^2 \leq [s,s][s',s'] = [s,s']^2 + [t,t]([s',s'] - 2\lambda[s,s'] + \lambda^2[s,s]) \leq [s,s']^2,$$

implying that $[t,t]([s',s'] - 2\lambda[s,s'] + \lambda^2[s,s]) = 0$. If $[t,t] = 0$, then $t = t' = 0$, and from the strict convexity of S we get that there is a real $\mu > 0$ with $s' = \mu s$. For this μ we have also $s' + t' = \mu(s+t)$. So we can assume that $[t,t] \neq 0$, and thus both $[s,s][s',s'] = [s,s']^2$ and $([s',s'] - 2\lambda[s,s'] + \lambda^2[s,s]) = 0$ hold. But S is a strictly convex space. Therefore, again for a nonzero s there is a real $\mu > 0$ with $s' = \mu s$. But this also implies $0 = (\mu - \lambda)^2[s,s]$, showing that $\mu = \lambda$ and $s' + t' = \mu(s+t)$. Using Lemma 3.2.3, we get the strict convexity of the embedding normed space.

Let now F be a linear isometry of H. It is clear that the linear operator J transforms the Minkowski product into the s.i.p. of the embedding space. Precisely we have $[v,w]^+ = [v, Jw]^-$. Now using the existence of the adjoint operator, the calculation

$$[v, Jw]^- = [v,w]^+ = [Fv, Fw]^+ = [Fv, JFw]^- = [v, F^T JFw]^-$$

holds for each pair of vectors v and w. But the embedding space is a non-degenerate one; thus we get the equality $J = F^T JF$, or equivalently $F^{-1} = JF^T J$. By its definition the last condition on F also holds.

Conversely, if F is a linear mapping satisfying the condition of the theorem, then it preserves the Minkowski product. In fact,

$$[Fv, Fw]^+ = [Fv, JFw]^- = [v, F^T JFw]^- = [v, Jw]^- = [v,w]^+.$$

It takes the hyperboloid H homeomorphically onto itself, implying that it takes a sheet onto a sheet. Our last condition guarantees that $F(H^+) = H^+$ and F is a linear isometry of H^+ as we stated.

We also reformulates the length of a path as follows. The Minkowski-Finsler semi-metric on H^+ is the function ds^2 which assigns at each point $v \in H^+$ the Minkowski product which is the restriction of the Minkowski product to the tangent space T_v. This positive Minkowski product varies differentiably with v. Let $U \leq V$ be a subspace and consider a map $f : U \longrightarrow V$. If it is a totally differentiable map (with respect to the norm of the embedding n-space in the sense of Frechet) then $f(T_v) = T_{f(v)}$ for the tangent spaces at v and $f(v)$, respectively and one can define the pullback semi-metric $f^\star(ds^2)$ at the point v by the following formula:

$$f^\star(ds^2)_v(u_1, u_2) = ds^2_{f(v)}(Df(u_1), Df(u_2)) = [Df(u_1), Df(u_2)]^+_{f(v)}.$$

The square root ds of the semi-metric function defined by $\sqrt{ds^2_v(u,u)}$ is the so called length element and the length of a path is the integral of the pullback length element by the differentiable map $c : \mathbb{R} \longrightarrow V$. This implies that if a linear isometry leaves the Minkowski-Finsler semi-metric invariant by the pullback, then it preserves the integrand, and thus preserves the

integral as well. Let now F be a linear isomorphism, and its restriction to H^+ be f. Compute the pullback metric as follows:

$$f^\star(ds^2)_v(u_1,u_2) = ds^2_{f(v)}(Df(u_1),Df(u_2)) = [Df(u_1),Df(u_2)]^+_{f(v)} =$$
$$= [DF(u_1),DF(u_2)]^+_{F(v)} = [F(u_1),F(u_2)]^+_{F(v)}$$

because F is linear. But it preserves the Minkowski product, and therefore we conclude that

$$[F(u_1),F(u_2)]^+_{F(v)} = [u_1,u_2]^+_v = (ds^2)_v(u_1,u_2).$$

This proves that a linear isometry of H^+ is also a topological and Finsler isometry on it. Finally, in a Finsler space a function preserving the distance transforms geodesics to geodesics (see in [21]). In our case this is also true, since it is basically determined by the definition of the distance and the smoothness properties which are the same in both cases. Since our space is homogeneous and linear isometry preserves the distance by the above argument, we can assume that $a = e_n$. Let now $b \neq a$ and consider the 2-plane $\langle a,b\rangle$ spanned by the vectors a and b. The restriction of the s.i.i.p. to the plane $\langle a,b\rangle$ is an i.i.p.; thus the restricted Finsler function is a Riemannian one. So the intersection $H \cap \langle a,b\rangle$ is a hyperbola in the embedding Euclidean 2-space. Thus we can parameterize the points of a path from a to b by $c(t) = sh(\tau)e + ch(t)e_n$ for $t \in [0,1]$ with $c(0) = a$ and $c(1) = b$. The length of an arc from 0 to x is

$$\int_0^x \sqrt{ch^2(\tau) - sh^2(\tau)}d\tau = x,$$

showing that the points of this arc satisfy the triangle inequality with equality. Consequently it is a geodesic on H^+ and therefore its arc-length is the distance of the points a and $c(x)$. On the other hand, we also have

$$[a,b]^+ = [e_n, sh(1)e + ch(1)e_n]^+ = [e_n, ch(1)e_n] = -ch(1) = -ch(d(a,c(1)) = -ch(d(a,b)).$$

□

As it can be seen from the formula in this theorem, the generalized adjoint of a linear isometry is a linear transformation. We also note that Theorem 3.2.4 in the i.p. case gives the characterization of the isometries of the hyperbolic space of dimension $(n-1)$.

3.2.2. Premanifolds in a generalized space-time model. There is no and we did not give a formal definition of an object calling in our work [9] by *premanifold*. We use this word for a set if it has a manifold-like structure with high freedom in the choosing of the distance function of its tangent hyperplanes. For example we get premanifolds if we investigate the hypersurfaces of a generalized space-time model. The most important types of manifolds as Riemannian, Finslerian or semi-Riemannian can be investigated in this way. The structure of our embedding space was introduced in [8] and in the next paper [9] we continued our investigations by the building up of differential geometry of hypersurfaces. We gave the pre-version of the usual semi-Riemannian or Finslerian spaces, the hyperbolic space, the de Sitter sphere, the light cone and the unit sphere of the rounding semi inner product space, respectively. In the case, when the space-like component of the generalized space-time model is a continuously differentiable semi inner product space then we get back the known and usable geometrical information on the corresponding hypersurfaces of a pseudo-Euclidean space, e.g. we showed that a pre-hyperbolic space has constant negative curvature.
Let F be a hypersurface defined by the function $f : S \longrightarrow V$. Here $f(s) = s + \mathfrak{f}(s)e_n$ denotes the point of F. The curve $c : \mathbb{R} \longrightarrow S$ define a curve on F. We assume that c is a C^2-curve.

DEFINITION 3.2.6 ([126]). *We say that a hypersurface is* convex *if it lies on one side of its each tangent hyperplanes. It is* strictly convex *if it is convex and its tangent hyperplanes contain precisely one points of the hypersurface, respectively.*

If we have a map $f : S \longrightarrow V$ then it can be decomposed to a sum of its space-like and time-like components. We have $f = f_S + f_T$ where $f_S : S \longrightarrow S$ and $f_T : S \longrightarrow T$, respectively. With respect to the embedding s.i.p space we can compute its Frechet derivative by $Df = [Df_S, Df_T]^T$ implying that $Df(s) = Df_S(s) + Df_T(s)$. For brevity introduce the following notation

$$[f_1(c(t)), \cdot]^{+\prime}_{D(f_2\circ c)(t)}(f_2(c(t))) := \left([(f_1)_S(c(t)), \cdot]'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t))) - (f_1)_T(c(t))((f_2)_T \circ c)'(t)\right).$$

Now we state:

LEMMA 3.2.4 ([9]). *If $f_1, f_2 : S \longrightarrow V$ are two C^2 maps and $c : \mathbb{R} \longrightarrow S$ is an arbitrary C^2 curve then*

$$([(f_1 \circ c)(t)), (f_2 \circ c)(t))]^+)' = [D(f_1 \circ c)(t), (f_2 \circ c)(t))]^+ + [(f_1 \circ c)(t)), \cdot]^{+\prime}_{D(f_2\circ c)(t)}((f_2 \circ c)(t)).$$

PROOF. By definition

$$([f_1 \circ c, f_2 \circ c)]^+)'|_t := \lim_{\lambda\to 0} \frac{1}{\lambda}\left([f_1(c(t+\lambda)), f_2(c(t+\lambda))]^+ - [f_1(c(t)), f_2(c(t))]^+\right) =$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\left([(f_1)_S(c(t+\lambda)), (f_2)_S(c(t+\lambda))] - [(f_1)_S(c(t)), (f_2)_S(c(t))]\right) +$$

$$+ \lim_{\lambda\to 0} \frac{1}{\lambda}\left([(f_1)_T(c(t+\lambda)), (f_2)_T(c(t+\lambda))] - [(f_1)_T(c(t)), (f_2)_T(c(t))]\right).$$

We prove that the first part is

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\left([(f_1)_S(c(t+\lambda)) - (f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] + [(f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] -\right.$$

$$\left. -[(f_1)_S(c(t)), (f_2)_S(c(t))]\right) = [D((f_1)_S \circ c)|_t, (f_2)_S(c(t))] + [(f_1)_S(c(t)), \cdot]'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t))).$$

To this take a coordinate system $\{e_1, \cdots, e_{n-1}\}$ in S and consider the coordinate-wise representation $(f_2)_S \circ c = \sum_{i=1}^{n-1}((f_2)_S \circ c)_i e_i$. Using Taylor's theorem for the coordinate functions we have that there are real parameters $t_i \in (t, t+\lambda)$, for which

$$((f_2)_S \circ c)(t+\lambda) = ((f_2)_S \circ c)(t) + \lambda D((f_2)_S \circ c)(t) + \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1}((f_2)_S \circ c)''_i(t_i)e_i.$$

Thus we can get

$$[(f_1)_S(c(t)), (f_2)_S(c(t+\lambda))] - [(f_1)_S(c(t)), (f_2)_S(c(t))] =$$

$$= [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S\circ c)(t)\lambda + \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1}((f_2)_S\circ c)''_i(t_i)e_i] - [(f_1)_S(c(t)), (f_2)_S(c(t))] =$$

$$= ([(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda] - [(f_1)_S(c(t)), (f_2)_S(c(t))]) +$$

$$+ [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda + \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1}((f_2)_S \circ c)''_i(t_i)e_i] -$$

$$- [(f_1)_S(c(t)), (f_2)_S(c(t)) + D((f_2)_S \circ c)(t)\lambda].$$

In the second argument of this product, the Lipschwitz condition holds with a real constant K for enough small λ's, so we have that the absolute value of the substraction of the last two terms is less or equal to

$$K\left[(f_1)_S(c(t)), \frac{1}{2}\lambda^2 \sum_{i=1}^{n-1}((f_2)_S \circ c)''_i(t_i)e_i\right].$$

Applying now the limit procedure at $\lambda \to 0$ we get the required equality.
In the second part $(f_1)_T$ and $(f_2)_T$ are real-real functions, respectively so

$$\lim_{\lambda\to 0} \frac{1}{\lambda}([(f_1)_T(c(t+\lambda)), (f_2)_T(c(t+\lambda))] - [(f_1)_T(c(t)), (f_2)_T(c(t))]) =$$

$$= -((f_1)_T \circ c)'(t)(f_2)_T(c(t)) - (f_1)_T(c(t))((f_2)_T \circ c)'(t).$$

Hence we have

$$([(f_1 \circ c)(t)), (f_2 \circ c)(t))]^+)' =$$
$$= [D((f_1)_S \circ c)(t), ((f_2)_S \circ c)(t))] + [(f_1)_S(c(t)), \cdot]'_{D((f_2)_S \circ c)(t)}(((f_2)_S \circ c)(t))) -$$
$$-((f_1)_T \circ c)'(t)(f_2)_T(c(t)) - (f_1)_T(c(t))((f_2)_T \circ c)'(t) =$$
$$= [D(f_1 \circ c)(t), f_2(c(t))]^+ \left([(f_1)_S(c(t)), \cdot]'_{D((f_2)_S \circ c)(t)}((f_2)_S(c(t))) - (f_1)_T(c(t))((f_2)_T \circ c)'(t)\right),$$

and the statement is proved. □

In an Euclidean space the *first fundamental form* is a positive definite quadratic form induced by the inner product of the tangent space. In generalized space-time model the first fundamental form is giving by the scalar square of the tangent vectors with respect to the Minkowski product restricted to the tangent hyperplane.

DEFINITION 3.2.7 ([9]). *The* first fundamental form *in a point* $(f(c(t))$ *of the hypersurface* F *is the product*

$$\mathrm{I}_{f(c(t)} := [D(f \circ c)(t), D(f \circ c)(t)]^+.$$

The variable of it is a tangent vector, a tangent vector of a variable curve c lying on F through the point $(f(c(t))$. We can see that the first fundamental form is homogeneous of the second order but (in general) it has no a bilinear representation.
In fact, by the definition of f, (if $\{e_i : i = 1 \cdots n-1\}$ is a basis in S) the computation

$$\mathrm{I}_{f(c(t))} = [\dot{c}(t) + (\mathfrak{f} \circ c)'(t)e_n, \dot{c}(t) + (\mathfrak{f} \circ c)'(t)e_n]^+ = [\dot{c}(t), \dot{c}(t)] - [(\mathfrak{f} \circ c)'(t)]^2 = [\dot{c}(t), \dot{c}(t)] -$$
$$- \sum_{i,j=1}^{n-1} \dot{c}_i(t)\dot{c}_j(t)\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t)) = [\dot{c}(t), \dot{c}(t)] - \dot{c}(t)^T \left[\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t))\right]_{i,j=1}^{n-1} \dot{c}(t)$$

shows that it is not a quadratic form. It would be a quadratic form if and only if the quantity

$$[\dot{c}(t), \dot{c}(t)] - \dot{c}(t)^T\dot{c}(t) = [\dot{c}(t), \dot{c}(t)] - \sum_{i=1}^{n-1} \dot{c}_i^2(t)$$

vanishes. Thus if the Minkowski product is an i.p. than we can assume that the basis $\{e_i\}$ in S is orthonormal and we have that the mentioned difference is vanishing, furthermore $c_i(t) = \langle e_i, c(t)\rangle = \langle c(t), e_i\rangle$ and $\dot{c}(t) = \sum_{i=1}^{n-1} \dot{c}_i(t)e_i$. So

$$\mathrm{I}_{f(c(t))} = \dot{c}(t)^T \left(\mathrm{Id} - \left[\mathfrak{f}'_{e_i}(c(t))\mathfrak{f}'_{e_j}(c(t))\right]_{i,j=1}^{n-1}\right) \dot{c}(t),$$

and we get back the classical local quadratic representation of the first fundamental form. Now if $c_i(t) = 0$ for $i \geq 3$ then $\det I = 1 - (\mathfrak{f}'_{e_1}(c(t)))^2 - (\mathfrak{f}'_{e_2}(c(t)))^2$.
We now extend the definition of the second fundamental form take into consideration that the product has neither symmetry nor bilinearity properties. If v is a tangent vector and n is a normal vector of the hypersurface at its point $f(c(t))$ then we have $0 = [v, n]^+ = [D(f \circ c)(t), (f \circ c)(t)]^+$. Using Lemma 3.2.4 and the notation follows it, we get $0 = ([D(f \circ c)(t), (n \circ c)(t)]^+)' = [D^2(f \circ c), n(c(t))]^+ + [D(f \circ c)(t), \cdot]^{+\prime}_{D(n \circ c)(t)}(n(c(t)))$.
We introduced the unit normal vector fields n^0 by the definition

$$n^0(c(t)) := \begin{cases} n(c(t)) & \text{if } n \text{ is a light-like vector} \\ \frac{n(c(t))}{\sqrt{|[n(c(t)), n(c(t))]^+|}} & \text{otherwise.} \end{cases}$$

DEFINITION 3.2.8 ([9]). *The* second fundamental form *at the point* $f(c(t))$ *defined by one of the equivalent formulas:*

$$\mathrm{II} := [D^2(f \circ c)(t), (n^0 \circ c)(t)]^+_{(f \circ c)(t)} = -[D(f \circ c)(t), \cdot]^{+\prime}_{D(n^0 \circ c)(t)}((n^0 \circ c)(t)).$$

By the structure of the generalized space-time model assuming that $n(s) = s + \mathfrak{n}(s)e_n$ we get that

$$\mathrm{II} = [D^2(f\circ c)(t), (n^0\circ c)(t)]^+_{(f\circ c)(t)} = \left[D(\dot c(t) + D(\mathfrak{f}\circ c)(t)e_n), \frac{c(t)+(\mathfrak{n}\circ c)(t)e_n}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]^+ =$$

$$= \frac{\left[\ddot c(t) + \left(\dot c(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot c(t) + \left[\mathfrak{f}'_{e_i}|_{c(t)}\right]\ddot c(t)\right)e_n, c(t)+\mathfrak{n}(c(t))e_n\right]^+}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= \frac{\left[\ddot c(t) + [\mathfrak{f}'_{e_i}|_{c(t)}]\ddot c(t)e_n, (n\circ c)(t)\right]^+ - \left(\dot c(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot c(t)\right)(\mathfrak{n}(c(t)))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= \frac{\left[D(f)|_{c(t)}\ddot c(t), (n\circ c)(t)\right]^+ - \left(\dot c(t)^T\left[\mathfrak{f}''_{e_i,e_j}|_{c(t)}\right]\dot c(t)\right)(\mathfrak{n}(c(t)))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}} =$$

$$= -\left(\dot c(t)^T\left[\frac{\mathfrak{f}''_{e_i,e_j}|_{c(t)}\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]^{n-1}_{i,j=1}\dot c(t)\right).$$

We now can adopt a determinant of this fundamental form. It is the determinant of its quadratic form:

$$\det \mathrm{II} := \det\left(\left[\frac{\mathfrak{f}''_{e_i,e_j}|_{c(t)}\mathfrak{n}(c(t))}{\sqrt{|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}}\right]^{n-1}_{i,j=1}\right).$$

If we consider a two-plane in the tangent hyperplane then it has a two dimensional pre-image in S by the regular linear mapping Df. The getting plane is a normed one and we can consider an Auerbach basis $\{e_1, e_2\}$ in it.

Definition 3.2.9 ([9]). *The* sectional principal curvature *of a 2-section of the tangent hyperplane in the direction of the* 2*-plane spanned by* $\{u = Df(e_1)$ *and* $v = Df(e_2)\}$ *are the extremal values of the function*

$$\rho(D(f\circ c)) := \frac{\mathrm{II}_{f\circ c(t)}}{\mathrm{I}_{f\circ c(t)}},$$

of the variable $D(f\circ c)$*. We denote them by* $\rho(u,v)_{\max}$ *and* $\rho(u,v)_{\min}$*, respectively. The* sectional (Gauss) curvature $\kappa(u,v)$ *(at the examined point* $c(t)$*) is the product*

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+\rho(u,v)_{\max}\rho(u,v)_{\min}.$$

In the case of a symmetric and bilinear product, both of the fundamental forms are quadratic and the sectional principal curvatures attained in orthogonal directions. They are the eigenvalues of the pair of quadratic forms $\mathrm{II}_{f\circ c(t)}$ and $\mathrm{I}_{f\circ c(t)}$. This implies that $\rho(u,v)_{\max}$ and $\rho(u,v)_{\min}$ are the solutions of the equality:

$$0 = \det\left(\mathrm{II}_{f\circ c(t)} - \lambda \mathrm{I}_{f\circ c(t)}\right) = \det\left(\mathrm{I}_{f\circ c(t)}\right)\det\left((\mathrm{I}_{f\circ c(t)})^{-1}\mathrm{II}_{f\circ c(t)} - \lambda \mathrm{Id}\right),$$

showing that

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+\rho(u,v)_{\max}\rho(u,v)_{\min} =$$

$$= [n^0(c(t)), n^0(c(t))]^+\det\left(\mathrm{I}^{-1}_{f\circ c(t)}\mathrm{II}_{f\circ c(t)}\right) = [n(c(t)), n(c(t))]^+\frac{\det \mathrm{II}_{f\circ c(t)}}{\det \mathrm{I}_{f\circ c(t)}} =$$

$$= [n^0(c(t)), n^0(c(t))]^+\frac{\left(\mathfrak{f}''_{e_1,e_1}|_{c(t)}\mathfrak{f}''_{e_2,e_2}|_{c(t)} - \left(\mathfrak{f}''_{e_1,e_2}|_{c(t)}\right)^2\right)(\mathfrak{n}(c(t)))^2}{\left(1-(\mathfrak{f}'_{e_1}(c(t)))^2-(\mathfrak{f}'_{e_2}(c(t)))^2\right)|[c(t),c(t)]-(\mathfrak{n}(c(t)))^2|}.$$

But we can choose for the function n

$$n(c(t)) := \mathfrak{f}'_{e_1}(c(t))e_1 + \mathfrak{f}'_{e_2}(c(t))e_2 + e_n$$

with $\mathfrak{n}(c(t)) = 1$ and for a 2-plane of the tangent hyperplane which contains only space-like vectors and has time-like normal vector with absolute value

$$[n(c(t)), n(c(t))]^+ = \sqrt{1 - (\mathfrak{f}'_{e_1}(c(t)))^2 - (\mathfrak{f}'_{e_2}(c(t)))^2}$$

getting the well-known formula

$$\kappa(u, v) = \frac{-\mathfrak{f}''_{e_1,e_1}|_{c(t)}\mathfrak{f}''_{e_2,e_2}|_{c(t)} + \left(\mathfrak{f}''_{e_1,e_2}|_{c(t)}\right)^2}{\left(1 - (\mathfrak{f}'_{e_1}(c(t)))^2 - (\mathfrak{f}'_{e_2}(c(t)))^2\right)^2}$$

(see in [49] p.95.).
The Ricci curvature of a Riemannian hypersurface at a point $p = (f \circ c)(t)$ in the direction of the tangent vector $v = D(f \circ c)$ is the sum of the sectional curvatures in the directions of the planes spanned by the tangent vectors v and u_i, where u_i are the vectors of an orthonormal basis of the orthogonal complement of v. This value is independent from the choosing of the basis. Choose random (by uniform distribution) the orthonormal basis! The corresponding sectional curvatures $\kappa(u_i, v)$ will be random variables with the same expected values. The sum of them is again a random variable which expected value corresponding to the Ricci curvature at p with respect to v. Hence it is equal to $n-2$-times the expected value of the random sectional curvature determined by all of the two planes through v. Similarly the scalar curvature of the hypersurface at a point is the sum of the sectional curvatures defined by any two vectors of an orthonormal basis of the tangent space, it is also can be considered as an expected value. This motivates the following definition:

DEFINITION 3.2.10 ([9]). *The* Ricci curvature $\mathrm{Ric}(v)$ *in the direction of the tangent vector* v *at the point* $f(c(t))$ *is*

$$\mathrm{Ric}(v)_{f(c(t))} := (n-2) \cdot E(\kappa_{f(c(t))}(u, v))$$

where $\kappa_{f(c(t))}(u, v)$ *is the random variable of the sectional curvatures of the two planes spanned by* v *and a random* u *of the tangent hyperplane holding the equality* $[u, v]^+ = 0$. *We also say that the* scalar curvature *of the hypersurface* f *at its point* $f(c(t))$ *is*

$$\Gamma_{f(c(t))} := \binom{n-1}{2} \cdot E(\kappa_{f(c(t))}(u, v)).$$

In [9] we investigated four special hypersurfaces as premanifold the pre-versions of the hyperbolic space, the de Sitter sphere, the light cone and the unit sphere of the rounding semi inner product space, respectively.
We examined the imaginary unit sphere as the set H^+.
The set G is the collection of those points of a generalized space-time model which has scalar square equal to one. In a pseudo-Euclidean space this set was called the *de Sitter sphere*. The tangent hyperplanes of the de Sitter sphere are pseudo-Euclidean spaces. G is not a hypersurface but we can restrict our investigation to the positive part of G defined by

$$G^+ = \{s + t \in G : t = \lambda e_n \text{ where } \lambda > 0\}.$$

We remark that the local geometries of G^+ and G topologically identical. G^+ is a hypersurface defined by the function $g(s) = s + \mathfrak{g}(s)e_n$, where $\mathfrak{g}(s) = \sqrt{-1 + [s, s]}$ for $[s, s] > 1$.
Let L^+ be the positive part of the double cone determined by the function: $l(s) = s + \sqrt{[s, s]}e_n$.
Finally the set K collects the points of the unit sphere of the embedding s.i.p. space. In a pseudo-Euclidean space it is the unit sphere of the embedding Euclidean space. Its tangent hyperplanes are pseudo-Euclidean spaces. K is not a hypersurface but we can also restrict our investigation to its positive part defined by $K^+ = \{s + t \in K : t = \lambda e_n \text{ where } \lambda > 0\}$. It can be defined by the function: $k(s) = s + \mathfrak{k}(s)e_n$, where $\mathfrak{k}(s) = \sqrt{1 - [s, s]}$ for $[s, s] < 1$.
The differential geometric properties of these four premanifolds are:

THEOREM 3.2.5 ([9]). *Let* H^+, G^+, L^+ *and* K^+ *denote the imaginary unit sphere, the de Sitter sphere, the light cone and the unit sphere of the embedding s.i.p. space, respectively.*

(1) *H^+ is always convex. It is strictly convex if and only if the s.i.p. space S is a strictly convex space.*
(2) *If S is a continuously differentiable s.i.p. space then H^+ has constant negative curvature.*
(3) *G^+ and its tangent hyperplanes are intersecting, consequently there is no point at which G would be convex.*
(4) *The de Sitter sphere G has constant positive curvature if S is a continuously differentiable s.i.p space.*
(5) *The light cone L^+ has zero curvatures if S is a continuously differentiable s.i.p space.*
(6) *K^+ is convex. If S is a strictly convex space, then K^+ is also strictly convex.*
(7) *The fundamental forms of K are*
   - $\mathrm{I} = [\dot{c},\dot{c}] - \frac{\left([\dot{c}(t),c(t)]+[c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1-[c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{1-[c(t),c(t)]}$,
   - $\mathrm{II} = \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left(-[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}\right) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\mathrm{I}$.

   The principal, sectional, Ricci and scalar curvatures at a point $k(c(t))$ are
   - $\rho_{\max}(u,v) = \rho_{\min}(u,v) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}$,
   - $\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+\rho(u,v)_{\max}\rho(u,v)_{\min} = \frac{1}{-1+2[c(t),c(t)]}$,
   - $\mathrm{Ric}(v)_{k(c(t))} := (n-2)\cdot E(\kappa_{k(c(t))}(u,v)) = \frac{n-2}{-1+2[c(t),c(t)]}$,
   - $\Gamma_{k(c(t))} := \binom{n-1}{2}\cdot E(\kappa_{f(c(t))}(u,v)) = \frac{\binom{n-1}{2}}{-1+2[c(t),c(t)]}$, *respectively.*

(8) *At the points of K^+ having the equality $2[c(t),c(t)] = 1$, all of the curvatures can be defined as in the case of the light cone and can be regarded as zero.*

PROOF. We prove these statements step by step.
(1) Let $w = s' + t'$ be a point of H^+ and consider the product

$$[w-v,v]^+ = [s'-s,s] + [t'-t,t] = [s',s] - [s,s] - (\lambda'-\lambda)\lambda = [s',s] - \lambda'\lambda + 1,$$

where $t' = \lambda' e_n$, $t = \lambda e_n$ and $s', s \in S$ with positive λ' and λ, respectively. Since

$$\sqrt{1+[s',s']} = \lambda' \text{ and } \sqrt{1+[s,s]} = \lambda$$

thus

$$[w-v,v]^+ = [s',s] - \sqrt{1+[s',s']}\sqrt{1+[s,s]} + 1 \le \sqrt{[s',s'][s,s]} - \sqrt{1+[s',s']}\sqrt{1+[s,s]} + 1 \le 0,$$

because of the relation $[s',s'][s,s] + 2\sqrt{[s',s'][s,s]} + 1 \le [s',s'][s,s] + ([s',s'] + [s,s]) + 1$. Remark that equality holds if and only if the norms of s' and s are equal to each other and thus $\lambda' = \lambda$, too. So we have $[s',s] - [s,s] = 0$, or equivalently $[s',s] = \sqrt{[s',s'][s,s]}$. From the characterization of the strict convexity of an s.i.p. space we get H^+ contains only the point v of the tangent space T_v if and only if the s.i.p. space S is strictly convex.
(2) To determine the first fundamental form consider the map $h = s + \mathfrak{h}(s)e_n$ giving the points of H^+. (Here $\mathfrak{h}(s) = \sqrt{1+[s,s]}$ is a real valued function.) Then we get that

$$\mathrm{I} = [\dot{c}(t) + (\mathfrak{h}\circ c)'(t)e_n, \dot{c}(t) + (\mathfrak{h}\circ c)'(t)e_n]^+ = [\dot{c}(t),\dot{c}(t)] - [(\mathfrak{h}\circ c)'(t)]^2,$$

where $\dot{c}(t)$ means the tangent vector of the curve c of S at its point $c(t)$. Using Lemma 3.2.1 and Theorem 3.2.3 we have

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\left([\dot{c}(t),c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1+[c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{1+[c(t),c(t)]}.$$

From this formula, by the Cauchy-Schwartz inequality, we can get a new proof for the fact that this form is positive. The second fundamental form of H^+ is

$$\mathrm{II} := [\ddot{c}(t) + (\mathfrak{h}\circ c)''(t)e_n, c(t) + (\mathfrak{h}\circ c)(t)e_n]^+_{(\mathfrak{h}\circ c)(t)} = [\ddot{c}(t),c(t)] - (\mathfrak{h}\circ c)''(t)\mathfrak{h}(c(t)),$$

since $n \circ c = h \circ c = c(t) + (\mathfrak{h} \circ c)(t)e_n$. First we compute the derivative of $(\mathfrak{h} \circ c)'(t) : \mathbb{R} \longrightarrow \mathbb{R}$ at its point t. We use again the formulas of Lemma 3.2.1 and Lemma 3.2.4 getting

$$(\mathfrak{h} \circ c)''(t) = ((\mathfrak{h} \circ c)')'(t) = \left(\frac{[\dot{c}(t), c(t)]}{\sqrt{1 + [c(t), c(t)]}}\right)' = \frac{[\dot{c}(t), c(t)]'}{\sqrt{1 + [c(t), c(t)]}} - \frac{\frac{[\dot{c}(t), c(t)]}{\sqrt{1+[c(t),c(t)]}}[\dot{c}(t), c(t)]}{(1 + [c(t), c(t)])}$$

and so

$$(\mathfrak{h} \circ c)''(t)\mathfrak{h}(c(t)) = [\dot{c}(t), c(t)]' - \frac{[\dot{c}(t), c(t)]^2}{1 + [c(t), c(t)]} = \left([\ddot{c}(t), c(t)] + [\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t))\right) - \frac{[\dot{c}(t), c(t)]^2}{1 + [c(t), c(t)]}.$$

Thus the second fundamental form is

$$\mathrm{II} = -[\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t), c(t)]^2}{1 + [c(t), c(t)]},$$

or using the formula

$$\|y\|\| \cdot \|''_{x,z}(y) = [x, \cdot]'_z(y) - \frac{\mathrm{Re}[x, y]\mathrm{Re}[z, y]}{\|y\|^2},$$

we get an equivalent form:

$$\mathrm{II} = -\|c(t)\|\| \cdot \|''_{\dot{c}(t),\dot{c}(t)}c(t) - \frac{[\dot{c}(t), c(t)]^2}{\|c(t)\|^2(1 + \|c(t)\|^2)}.$$

If we also assume that the norm is a C^2 function of its argument then we can use Theorem 3.2.3 and we get

$$\mathrm{II} = -[\dot{c}(t), \dot{c}(t)] + \frac{[\dot{c}(t), c(t)]^2}{1 + [c(t), c(t)]} = -\mathrm{I}.$$

By the positivity of the first fundamental form on H^+, we get that the second fundamental form is negative definite and

$$\rho(u, v)_{\max} = \rho(u, v)_{\min} = -1.$$

This implies that the sectional curvatures are equal to -1, the Ricci and scalar curvatures in any direction at any point is $-(n-2)$ and $-\binom{n-1}{2}$, respectively.

(3) At an arbitrary point of G^+ there are two sets lying on G^+ and having in distinct halfspaces with respect to the corresponding tangent hyperplane. The first set is the intersection of the 2-plane spanned by e_n and $s + t \in M$; and the other one is an arbitrary curve of the $(n-2)$-hypersurface defined by the intersection of G and the hyperplane $S + (s + t)$. In fact, a normal vector of the tangent hyperplane at $s + t$ is itself $s + t$, because we have

$$\left[e + \frac{[e, s]}{\sqrt{-1 + [s, s]}}e_n, s + \sqrt{-1 + [s, s]}e_n\right]^+ = 0.$$

Thus with $\alpha > \frac{1}{\sqrt{[s,s]}}$ we have

$$\left[\left(\alpha s + \sqrt{-1 + [\alpha s, \alpha s]}e_n\right) - \left(s + \sqrt{-1 + [s, s]}e_n\right), s + \sqrt{-1 + [s, s]}e_n\right]^+ =$$

$$= (\alpha - 1)[s, s] + (\sqrt{-1 + [s, s]} - \sqrt{-1 + [\alpha s, \alpha s]})\sqrt{-1 + [s, s]} =$$

$$= -1 + \alpha[s, s] - \sqrt{(-1 + [\alpha s, \alpha s])(-1 + [s, s])} =$$

$$= \alpha[s, s] - 1 - \sqrt{1 - (1 + \alpha^2)[s, s] + \alpha^2[s, s]^2} \geq 2(\alpha[s, s] - 1) > 2(\|s\| - 1) \geq 0.$$

On the other hand if $s' + t \in M$ arbitrary, then $\|s'\| = \|s\|$ thus $[s' - s + (t - t), s + t]^+ = [s', s] - [s, s] \leq \sqrt{[s', s']}\sqrt{[s, s]} - [s, s] = 0$, with equality if and only if $s' = \pm s$.

(4) Using the function g, the first fundamental form has the form

$$\mathrm{I} = [\dot{c}(t) + (\mathfrak{g} \circ c)'(t)e_n, \dot{c}(t) + (\mathfrak{g} \circ c)'(t)e_n]^+ = [\dot{c}(t), \dot{c}(t)] - [(\mathfrak{g} \circ c)'(t)]^2.$$

Using Lemma 3.2.1 and Theorem 3.2.3 we get

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\Big([\dot{c}(t),c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\Big)^2}{4(-1+[c(t),c(t)])} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

Furthermore we also have that $n\circ c = g\circ c = c(t) + (\mathfrak{g}\circ c)(t)e_n$ thus we get:

$$\mathrm{II} := [\ddot{c}(t) + (\mathfrak{g}\circ c)''(t)e_n, c(t) + (\mathfrak{g}\circ c)(t)e_n]^+_{(\mathfrak{g}\circ c)(t)} = [\ddot{c}(t),c(t)] - (\mathfrak{g}\circ c)''(t)\mathfrak{g}(c(t)).$$

The derivative of the real function $(\mathfrak{g}\circ c)'(t) = D(\mathfrak{g}\circ c)(t) : \mathbb{R}\longrightarrow\mathbb{R}$ at its point t is:

$$(\mathfrak{g}\circ c)''(t) = \frac{[\dot{c}(t),c(t)]'}{\sqrt{-1+[c(t),c(t)]}} - \frac{\frac{[\dot{c}(t),c(t)]}{\sqrt{-1+[c(t),c(t)]}}[\dot{c}(t),c(t)]}{(-1+[c(t),c(t)])}$$

so by Lemma 3.2.4

$$(\mathfrak{g}\circ c)''(t)\mathfrak{g}(c(t)) = [\dot{c}(t),c(t)]' - \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]} = \Big([\ddot{c}(t),c(t)] + [\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t))\Big) - \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

Thus we have

$$\mathrm{II} = -[\dot{c}(t),\cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}.$$

If we assume again that the norm is a C^2 function of its argument then we can use again Theorem 3.2.3 and we get

$$\mathrm{II} = -[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]} = -\mathrm{I},$$

as in the case of H^+. The principal curvatures are equal to -1. But the scalar squares of the normal vectors is positive at all points of G^+ implying that the sectional curvatures are equal to 1. The Ricci curvatures in any directions and at any points are equal to $(n-2)$, moreover the scalar curvatures at any points are equal to $\binom{n-1}{2}$.

(5) If S is a uniformly continuous s.i.p. space, then the tangent vectors at s are of the form:

$$u = \alpha\left(e + \|\cdot\|'_e(s)e_n\right) = \alpha\left(e + \frac{[e,s]}{\sqrt{[s,s]}}e_n\right).$$

Thus all tangents orthogonal to $l(s)$ which is also a tangent vector. (Choose $e = s^0$ and $\alpha = \|s\|$!) But the orthogonal companion of a neutral vector in a s.i.i.p space is an $(n-1)$-dimensional degenerated subspace containing it (Theorem 3.1.1), tangent hyperplanes are exist at every points of L^+ and it is an $(n-1)$-dimensional degenerated subspace of V. This also a support hyperplane of L. In fact, by $v = s+t$ and $w = s'+t'$ we get

$$[w-v,v]^+ = [s',s] + [t',t] = [s',s] - \lambda'\lambda$$

where $t' = \lambda' e_n$, $t = \lambda e_n$ and $s', s\in S$ with positive λ' and λ, respectively. Since $\sqrt{[s',s']} = \lambda'$ and $\sqrt{[s,s]} = \lambda$ thus $[w-v,v]^+ = [s',s] - \sqrt{[s',s']}\sqrt{[s,s]} \le 0$ holds. We remark that equality holds if and only if $s' = \alpha s$ meaning that there is only one line of L^+ in the tangent space T_v. Thus the light cone is convex and thus the second fundamental form is semi-definite quadratic form. It also follows that any other vectors of the tangent hyperplane are space-like ones and there are two types of tangent 2-planes; one of them space-like plane and the other one contains space-like vectors and a doubled line of light-like vectors. In the first case, the corresponding principal and sectional curvatures is well defined and have negative values, respectively. To determine it we compute the fundamental forms.

In the case when S is continuously differentiable, the first fundamental form is

$$\mathrm{I} = [\dot{c},\dot{c}] - \frac{\Big([\dot{c}(t),c(t)] + [c(t),\cdot]'_{\dot{c}(t)}(c(t))\Big)^2}{4[c(t),c(t)]} = [\dot{c},\dot{c}] - \frac{[\dot{c}(t),c(t)]^2}{[c(t),c(t)]},$$

and the second one is

$$\mathrm{II} = -[\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t), c(t)]^2}{[c(t), c(t)]} = -[\dot{c}(t), \dot{c}(t)] + \frac{[\dot{c}(t), c(t)]^2}{[c(t), c(t)]} = -\mathrm{I}.$$

Thus the principal curvatures are -1 as in the cases of the unit spheres. However our definition gives at such a point zero sectional curvature for it, because of the zero lengths of the normal vectors. The above computation can be used in the second case, too. Agreed that we calculate the fundamental forms only non-light-like directions, so on the plane of the second type the principal curvatures are also -1 and the sectional curvatures are zero, too. This implies that the Ricci and scalar curvatures are also zero, respectively.

(6) The directional derivatives of the function $\mathfrak{k} : s \longmapsto \sqrt{1-[s,s]}$ for $[s,s] < 1$ gives the corresponding tangent vectors of form $u = \alpha(e + \mathfrak{k}'_e(s)e_n)$. Since by the function $\mathfrak{f} : s \longmapsto \sqrt{1+[s,s]}$, we have the equality $\mathfrak{f}^2(s) + \mathfrak{k}^2(s) = 2$ the derivative in the direction of the unit vector $e \in S$ is $\mathfrak{k}'_e(s) = -\frac{[e,s]}{\sqrt{1-[s,s]}}$ meaning that $[u,u]^+ = \alpha^2\left(1 - \frac{[e,s]^2}{(1-[s,s])}\right) = \alpha^2\frac{1-[s,s]-[e,s]^2}{1-[s,s]}$. From this we can see immediately that

$$\begin{array}{ll} [u,u]^+ > 0 & \text{if } 1-[s,s] > [e,s]^2 \\ [u,u]^+ = 0 & \text{if } 1-[s,s] = [e,s]^2 \\ [u,u]^+ < 0 & \text{if } 1-[s,s] < [e,s]^2. \end{array}$$

It follows that the vector s' of the $(n-2)$-subspace of S orthogonal to s gives a space-like tangent vector and the vector corresponding to αs is a time-like one. Let $w = s' + t'$ be a point of K^+ and consider the product

$$[w - v, n_v]^+ = [s' - s, s''] + [t' - t, t''] = [s', s''] - [s, s''] - (\lambda' - \lambda)\lambda'',$$

where $t'' = \lambda'' e_n$, $t' = \lambda' e_n$, $t = \lambda e_n$ and $s'', s', s \in S$ with positive λ'', λ' and λ, respectively. Since $\sqrt{1-[s',s']} = \lambda'$ and $\sqrt{1-[s,s]} = \lambda$ and $n_v = s - \sqrt{1-[s,s]}e_n$ thus

$$[w-v, n_v]^+ = [s', s] + \sqrt{1-[s',s']}\sqrt{1-[s,s]} - 1 \leq \sqrt{[s',s'][s,s]} + \sqrt{1-[s',s']}\sqrt{1-[s,s]} - 1 \leq 0,$$

because $2\sqrt{[s',s'][s,s]} \leq [s',s'] + [s,s]$). We remark that equality holds in the inequalities if and only if the norms of s' and s are equal to each other. So we have the equality $[s',s] - [s,s] = 0$, or equivalently $[s',s] = \sqrt{[s',s'][s,s]}$. We also get that v is the only point of K^+ lying on the tangent space T_v if and only if the s.i.p. space S is strictly convex.

(7) Using the function k we get

$$\mathrm{I} = [\dot{c}(t), \dot{c}(t)] - [(\mathfrak{k} \circ c)'(t)]^2.$$

Using Lemma 3.2.1 and Theorem 3.2.3 we have

$$\mathrm{I} = [\dot{c}, \dot{c}] - \frac{\left([\dot{c}(t), c(t)] + [c(t), \cdot]'_{\dot{c}(t)}(c(t))\right)^2}{4(1 - [c(t), c(t)])} = [\dot{c}, \dot{c}] - \frac{[\dot{c}(t), c(t)]^2}{1 - [c(t), c(t)]},$$

and assuming that $2[c(t), c(t)] \neq 1$ we get

$$\mathrm{II} = \left[\ddot{c}(t) + (\mathfrak{k} \circ c)''(t)e_n, \frac{c(t) - (\mathfrak{k} \circ c)(t)e_n}{\sqrt{|-1 + 2[c(t), c(t)]|}}\right]^+_{(\mathfrak{k} \circ c)(t)} = \frac{1}{\sqrt{|-1 + 2[c(t), c(t)]|}}\left([\ddot{c}(t), c(t)] + (\mathfrak{k} \circ c)''(t)\mathfrak{k}(c(t))\right).$$

Lemma 3.2.4 implies that

$$(\mathfrak{k} \circ c)''(t)\mathfrak{k}(c(t)) = -[\dot{c}(t), c(t)]' + \frac{[\dot{c}(t), c(t)]^2}{1 - [c(t), c(t)]} = -\left([\ddot{c}(t), c(t)] + [\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t))\right) + \frac{[\dot{c}(t), c(t)]^2}{1 - [c(t), c(t)]}.$$

thus we have

$$\mathrm{II} = \frac{1}{\sqrt{|-1 + 2[c(t), c(t)]|}}\left(-[\dot{c}(t), \cdot]'_{\dot{c}(t)}(c(t)) + \frac{[\dot{c}(t), c(t)]^2}{1 - [c(t), c(t)]}\right).$$

Assuming that S is continuously differentiable and using Theorem 3.2.3 we get

$$\mathrm{II} = \frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\left(-[\dot{c}(t),\dot{c}(t)] + \frac{[\dot{c}(t),c(t)]^2}{-1+[c(t),c(t)]}\right) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}\mathrm{I}.$$

The principal curvatures at a point $k(c(t))$ are

$$\rho_{\max}(u,v) = \rho_{\min}(u,v) = -\frac{1}{\sqrt{|-1+2[c(t),c(t)]|}}$$

giving the sectional curvatures

$$\kappa(u,v) := [n^0(c(t)), n^0(c(t))]^+ \rho(u,v)_{\max}\rho(u,v)_{\min} = \frac{1}{-1+2[c(t),c(t)]}.$$

The Ricci curvatures in any directions at the point $k(c(t))$ are equal to

$$\mathrm{Ric}(v)_{k(c(t))} := (n-2)\cdot E(\kappa_{k(c(t))}(u,v)) = \frac{n-2}{-1+2[c(t),c(t)]}$$

and the scalar curvature of the hypersurface K^+ at its point $k(c(t))$ is

$$\Gamma_{k(c(t))} := \binom{n-1}{2}\cdot E(\kappa_{f(c(t))}(u,v)) = \frac{\binom{n-1}{2}}{-1+2[c(t),c(t)]}.$$

(8) Finally we remark that at the points of K^+ having the equality $2[c(t),c(t)] = 1$ all of the curvatures can be defined as in the case of the light cone and can be regarded to zero. □

As we saw H^+ which is the generalization of the hyperbolic space can be considered as a premanifold it is the *pre-hyperbolic space* in our terminology. We can tell about G as a premanifold of constant positive curvature and we may say that it is a *pre-sphere* L is a premanifold with zero sectional, Ricci and scalar curvatures, respectively. We may also say that it is a *pre-Euclidean* space. K^+ is an example to a premanifold with non-constant curvatures.

3.3. The metric space of norms

The investigations of the author on the generalized space-time models of changing shape proposed that define "Gaussian" (or other type) probability measure on the metric space of centrally symmetric convex, compact bodies. This leads to a very important part of convex geometry to the investigation of the Space of Convex Bodies. A good survey on the long history can be found in Section 13 of the book [71] of P. Gruber. We shall investigate the probability space of norms defined on a real, n-dimensional Euclidean space V. A norm function on V defined by its unit ball K, which is a centrally symmetric in O convex body. Such bodies give a closed proper subset $\mathcal{K}_0$ of the space of convex bodies $\mathcal{K}$ of $(V, \langle\cdot,\cdot\rangle)$[1]. It is known that the Hausdorff distance δ^h is a metric on $\mathcal{K}$ and with this metric $(\mathcal{K},\delta^h)$ is a locally compact space. (See in [71],[72].) Thus there should be many measures available on these space. Unfortunately this is not so. Bandt and Baraki in [22] proved answering to a problem of McMullen [135] that there is no positive σ-finite Borel measure on it which is invariant with respect to all isometries of $(\mathcal{K},\delta^h)$ into itself. This result exclude the possibility of the existence of a volume-type measure. It was a natural question that can whether be found such a σ-finite Borel measure on $\mathcal{K}$ which holds the property that it is non-zero for any open set of $\mathcal{K}$ and invariant under rigid motions of the embedding vector space. This long standing question was answered in the last close by Hoffmann in [90]. His result can be summarized as follows. Each σ-finite rotation and translation invariant Borel measure on $(\mathcal{K},\delta^h)$ is the vague limit of such measures and that each σ-finite Borel measure on $(\mathcal{K},\delta^h)$ is the vague limit of measures of the form $\sum_{i=1}^{\infty}\alpha_n\delta_{K_n}$,

[1]We rather denote in this paper the space of O-symmetric convex bodies by $\mathcal{K}_0$ as the space of convex bodies with centroid O.

where $\{K_n\ , n \in \mathbb{N}\}$ is a countable, dense subset of $(\mathcal{K}, \delta^h)$, (α_n) is a sequence of positive real numbers for which $\sum_{i=1}^{\infty} \alpha_n < \infty$ and δ_{K_n} denote the Dirac measure concentrated at K_n.
Hoffmann also observed that a result of Bárány [23] "suggest that it might not be possible to define a "uniform" probability measure on the set of all polytopes which have rational vertices and are contained in the unit ball". The known concept of Gaussian random convex bodies [125] gives a poor class of Gaussian measures because of a random convex body is Gaussian if and only if there exists a deterministic body and a Gaussian random vector such that the random body is the sum of the deterministic one and the random vector almost surely. He asked "whether there exists an alternative approach to "Gaussian" random convex bodies which yields a richer class of "Gaussian" measures on $(\mathcal{K}, \delta^h)$.
Our observation is that on certain large probability space the uniformity or normality properties could be only "relative" one and thus we can require these properties in their impacts through a given function of the space. More precisely, we require the normality or uniformity on a pushforward measure by a given geometric function of the elements of the space (here on the space of convex bodies). To this purpose we use the thinness function $\alpha_0(K)$ of K defined by the help of the concepts of diameter $\mathrm{diam}K$ and width $\mathrm{w}(K)$ of a convex body K.

3.3.1. The thinness function and other definitions. We recall some necessary definitions. Deeper understanding of the subject on convex geometry and geometric measure theory I suggest to read the books [71], [94] and [133] where all properties of the following concepts can be found. Let $\mathcal{K}$ be the set of convex bodies of an Euclidean vector space of dimension n. It is endowed with the topology induced by the Hausdorff metric δ^h, which was defined in subsection 2.1.4. If we consider a topology on $\mathcal{K}$ or on a subspace of it, such as the space of O-symmetric convex bodies $\mathcal{K}_0$, it is always assumed that it is the topology induced by δ^h.
From geometric measure theory we will use the concepts of Borel, Dirac, Haar and Lebesgue-measure. All of these concepts can be found in [56] or [86]. We also use some basic tools of probability theory, e.g. the concepts of truncated Gaussian and uniform distributions, and the concept of the pushforward and pullback of a measure. The reader can read on these concept on the internet or in basic works on probability theory e.g. in [58] or [88].
Let denote by $\mathrm{w}(K)$ the infimum of the distances between parallel support hyperplanes of the convex body K. This is the *width* of K. The *diameter* of K ($\mathrm{diam}K$) is the supremum of the distances between two points of K. It can be regarded also as the supremum of the distances between parallel support hyperplanes of K. By these two quantities we defined a new one.

Definition 3.3.1 ([10]). *Let denote by $\alpha_0(K)$ the number*

$$\alpha_0(K) = \frac{\mathrm{diam}K}{\mathrm{w}(K) + \mathrm{diam}K}.$$

We call it the thinness *of the convex body K.*

The thinness is $\frac{1}{2}$ in the case of the Euclidean ball only and it is equal to 1 if K has of dimension less or equal to $(n-1)$.
Let now B_E be the unit ball of the embedding Euclidean space and let define the unit sphere of $\mathcal{K}_0$ around B_E by the equality: $\mathcal{K}_0^1 := \{K \in \mathcal{K}_0 \mid \delta^h(K, B_E) = 1\}$.
The following lemma shows the usable of the thinness function in our investigation.

Lemma 3.3.1 ([10]). *If $K \in \mathcal{K}_0^1$ and $\alpha_0 := \alpha_0(K)$ is the thinness of K then we have*

$$\delta^h(\alpha K, B_E) = \begin{cases} 2\alpha - 1 & \text{if} \quad \alpha_0 \leq \alpha \\ 2\alpha + 1 - 2\frac{\alpha}{\alpha_0} & \text{if} \quad 0 \leq \alpha < \alpha_0. \end{cases}$$

Proof. Assume that $\delta^h(K, B_E)$ is the distance of the points $x \in \mathrm{bd}\, B_E$ and $y \in \mathrm{bd}\, K$. Then $\|y\|_E = \|x\|_E + 1 = 2$ and $0, x, y$ are collinear. (We note that the norm of the point y is also

the half of the diameter diam$K$ of K with respect to the Euclidean metric.) This implies that for $\alpha > 1$ the points $\frac{1}{\alpha}x$ and y give a segment with length $\delta^h\left(K, \frac{1}{\alpha}B_E\right)$ and thus

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|y - \frac{1}{\alpha}x\right\|_E = \|y\|_E - \frac{1}{\alpha}\|x\|_E = 2 - \frac{1}{\alpha}$$

holds. If $\alpha < 1$ then the situation is a little bit more complicated. In this case there is a real number $\alpha_0 \in \left[\frac{1}{2}, 1\right)$ such that if $\alpha_0 \leq \alpha < 1$ then again

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|y - \frac{1}{\alpha}x\right\|_E = \|y\|_E - \frac{1}{\alpha}\|x\|_E = 2 - \frac{1}{\alpha}$$

but for $\alpha_0 \geq \alpha > 0$ we have a new pair of points $y' \in \mathrm{bd}\, K$ and $x' \in \mathrm{bd}\, B_E$ where the distance attained. The point y' is a point of $\mathrm{bd}\, K$ with minimal norm and we have the equality

$$\frac{1}{\alpha_0} - \|y'\| = 2 - \frac{1}{\alpha_0}.$$

Thus the norm of y' is equal to $2(\frac{1}{\alpha_0} - 1)$. In this case

$$\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \left\|-y' + \frac{1}{\alpha}x'\right\|_E = \frac{1}{\alpha} - 2\left(\frac{1}{\alpha_0} - 1\right) = 2 + \frac{1}{\alpha} - \frac{2}{\alpha_0}.$$

We thus have the equality

$$\delta^h(\alpha K, B_E) = \alpha\delta^h\left(K, \frac{1}{\alpha}B_E\right) = \begin{cases} 2\alpha - 1 & \text{if} \quad \alpha_0 \leq \alpha \\ 2\alpha + 1 - 2\frac{\alpha}{\alpha_0} & \text{if} \quad 0 \leq \alpha < \alpha_0. \end{cases}$$

The constant α_0 depends only on the body K and it has the following geometric meaning. $\|y'\|_E = \frac{2}{\alpha_0} - 2$ is the half of the width $\mathrm{w}(K)$ of the centrally symmetric body K, because it is a point on $\mathrm{bd}K$ with minimal norm. So we can see that

$$\frac{1}{2} \leq \alpha_0 = \frac{2}{\|y'\|_E + 2} = \frac{\mathrm{diam}K}{\mathrm{w}(K) + \mathrm{diam}K} < 1$$

as we stated. □

3.3.2. The constructed measure and its measure theoretic properties. We now construct a measure on $\mathcal{K}_0^1$ which pushforward by the thinness function has uniform distribution. To this (following Hoffmann's paper) we introduced the orbits of a body K about the special orthogonal group $SO(n)$ by $[K]$. These are compact subsets of $\mathcal{K}_0^1$, and if we consider an open subset of $\mathcal{K}_0^1$ then the union of the corresponding orbits is also open. Hence there exists a measurable mapping $s : \mathcal{K}_0^1 \longrightarrow \mathcal{K}_0^1$ such that $s(K) = s(K')$ if and only if K and K' are on the same orbit. Let $\widetilde{\mathcal{K}_0^1} := \{K \in \mathcal{K}_0^1\ ,\ s(K) = K\}$ which is measurable subset of $\mathcal{K}_0^1$. We equip it with the induced topology of $\mathcal{K}_0^1$. Finally let $\Phi_{2a}^1 : \widetilde{\mathcal{K}_0^1} \times SO(n) \longrightarrow \mathcal{K}_0^1$ is the mapping defined by the equality: $\Phi_{2a}^1(K, \Theta) = \Theta K$. Our notation is analogous with the notation of [90]. It was proved in [90] (Lemma 2) that a non-trivial σ-finite measure μ_0 on $\mathcal{K}_0$ is invariant under rotations (meaning that for $\Theta \in SO(n)$ we have $\mu_0(\mathcal{A}) = \mu(\Theta\mathcal{A})$ for all Borel sets $\mathcal{A}$ of $\mathcal{K}_0$) if and only if there exists a σ-finite measure $\widetilde{\mu_0}$ on $\widetilde{\mathcal{K}_0}$ such that $\mu_0 = \Phi_{2a}(\widetilde{\mu_0} \otimes \nu_n)$, where ν_n is the Haar measure on $SO(n)$. It is obvious that in the case of $\mathcal{K}_0^1$ there is a similar result by our mapping $\Phi_{2a}^1(K, \Theta)$ which is the restriction of Hoffmann's map $\Phi_{2a}(K, \Theta)$ onto the set $\mathcal{K}_0^1$. First we chose a countable system of bodies K_m to define a probability measure on $\widetilde{\mathcal{K}_0^1}$. Without loss of generality we may assume that each of the bodies of $\widetilde{\mathcal{K}_0^1}$ has a common diameter of length 4 denoted by d, which lies on the n^{th} axe of coordinates (hence it is the convex hull of the points $\{2e_n, -2e_n\}$). Consider the set of diadic rational numbers in $(0, 2]$. We can write them as follows:

$$\left\{m(n,k) := \frac{k}{2^n} \text{ where } n = 0, \cdots \infty \text{ and for a fixed } n,\ 0 < k \leq 2^{n+1}\right\}.$$

Define the body $K_{m(n,k)}$ as the convex hull of the union of the segment d and the ball around the origin with radius $m(n,k)$. For each n we have 2^{n+1} such bodies, thus the definition

$$\widetilde{\mu_0^1} := \lim_{n\to\infty} \sum_{k=1}^{2^{n+1}} \frac{1}{2^{n+1}} \delta_{K_{m(n,k)}}$$

define a probability measure on $\widetilde{\mathcal{K}_0^1}$.

LEMMA 3.3.2 ([10]). *The pushforward measure* $\mathrm{w}(K)^{-1}(\widetilde{\mu_0^1})$ *has uniform distribution on the interval* $(0,4]$.

PROOF. Let $B' = (0,x]$ be a level set of $(0,4]$. By definition

$$\mathrm{w}(K)^{-1}(\widetilde{\mu_0^1})(B') = \widetilde{\mu_0^1}\left(\left\{K \in \widetilde{\mathcal{K}_0^1} \mid \mathrm{w}(K) \in B'\right\}\right) = \lim_{n\to\infty} \sum_{\substack{K_{m(n,k)} \in \mathrm{w}(K)^{-1}(B') \\ 0<k\leq 2^{n+1}}} \frac{1}{2^{n+1}} =$$

$$= \lim_{n\to\infty} \sum_{2m(n,k)\in B'} \frac{1}{2^{n+1}} = \lim_{n\to\infty} \sum_{2m(n,k)<x} \frac{1}{2^{n+1}} = \lim_{n\to\infty} \sum_{k=1}^{2^{n-1}x} \frac{1}{2^{n+1}} = \frac{x}{4}$$

showing that $\mathrm{w}(K)^{-1}(\widetilde{\mu_0^1})$ is the uniform distribution of the interval $(0,4]$. □

The Gaussian measure γ of the n^2-dimensional matrix space $\mathbb{R}^{n\times n}$ defined by the density function $G(X)$

$$G(X)\mathrm{d}\lambda^{\mathrm{n}^2} := \frac{1}{\left(\sqrt{2\pi}\right)^{n^2}} e^{-\frac{1}{2}\mathrm{Tr}(X^T X)} \mathrm{d}\lambda^{\mathrm{n}^2},$$

where $\mathrm{d}\lambda^{\mathrm{n}^2}$ is the n^2-dimensional Lebesgue measure. The Haar measure ν_n of $\mathbb{R}^{n\times n}$ is the pushforward measure of the Gaussian measure by the mapping M defined by the Gram-Schmidt process (see in [107]). We stated the following:

THEOREM 3.3.1 ([10]). *Let define the measure* $\widetilde{\nu_0^1}$ *by density function* $\mathrm{d}\widetilde{\nu_0^1} = \frac{4}{(\mathrm{w}+4)^2}\mathrm{d}\widetilde{\mu_0^1}$. *Then*

$$\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)$$

is a probability measure with uniform distribution on $[\frac{1}{2},1)$.

PROOF. We are stating that the pushforward measure $\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)\right)$ has uniform distribution on $[\frac{1}{2},1)$ if and only if the pushforward measure $\mathrm{w}(K)^{-1}\left(\widetilde{\mu_0^1}\right)$ has uniform distribution on $(0,4]$. To prove this consider a Borel set B of $[\frac{1}{2},1)$ and its image B' under the bijective transformation $\tau : t \mapsto \tau(t) := \frac{4}{t} - 4$. Of course B' is a Borel set of the interval $(0,4]$ which is the image of $[\frac{1}{2},1)$ with respect to τ. We now have that

$$\int_B \mathrm{d}\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right) = \alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right)(B) =$$

$$= \Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)(\alpha_0(K)^{-1}(B)) = \widetilde{\nu_0^1}\left(\left(\Phi_{2a}^1\right)_1^{-1}\left((\alpha_0(K)^{-1}(B))\right)\right)\nu_n\left(\left(\Phi_{2a}^1\right)_2^{-1}\left(\alpha_0(K)^{-1}(B)\right)\right)$$

where $\left(\Phi_{2a}^1\right)_1^{-1}$ and $\left(\Phi_{2a}^1\right)_2^{-1}$ means the components of the set-valued inverse of the function Φ_{2a}^1, respectively. Since $\left(\Phi_{2a}^1\right)_2^{-1}(\alpha_0(K)^{-1}(B)))$ is the group $O(n)$ we have that

$$\int_B \mathrm{d}\alpha_0(K)^{-1}\left(\Phi_{2a}^1\left(\widetilde{\nu_0^1}\otimes\nu_n\right)\right) = \widetilde{\nu_0^1}\left(\left(\Phi_{2a}^1\right)_1^{-1}\left(\alpha_0(K)^{-1}(B)\right)\right) = \int_{\left(\Phi_{2a}^1\right)_1^{-1}\left(\alpha_0^{-1}(B)\right)} \mathrm{d}\widetilde{\nu_0^1}.$$

On the other hand

$$\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0^{-1}(B)\right) = \left\{\widetilde{K} \in \widetilde{\mathcal{K}^1_0} \mid \alpha_0(\widetilde{K}) = \frac{4}{\mathrm{w}(\widetilde{K})+4} \in B\right\} =$$

$$= \left\{\widetilde{K} \in \widetilde{\mathcal{K}^1_0} \mid \mathrm{w}(\widetilde{K}) \in B' = \frac{4}{B} - 4\right\}$$

implying that

$$\int\limits_{\left(\Phi^1_{2a}\right)^{-1}_1\left(\alpha_0^{-1}(B)\right)} \mathrm{d}\widetilde{\nu^1_0} = \int\limits_{\left\{\widetilde{K}\in\widetilde{\mathcal{K}^1_0}|\mathrm{w}(\widetilde{K})\in B'\right\}} \frac{4}{(w+4)^2}\mathrm{d}\widetilde{\mu^1_0},$$

and it is equal to

$$\int\limits_{\tau\in B'} \frac{4}{(4+\tau)^2}\mathrm{d}\tau = \int\limits_{t\in B} \mathrm{d}t$$

if and only if $\mathrm{w}(K)^{-1}\left(\widetilde{\mu^1_0}\right)$ has uniform distribution on $(0,4]$ as we stated.

Since Lemma 3.3.2 says that $\mathrm{w}(K)^{-1}\left(\widetilde{\mu^1_0}\right)$ has uniform distribution on the interval $[0,4]$ we also proved the theorem. □

Let denote by ν^1_0 the measure $\Phi^1_{2a}\left(\widetilde{\nu^1_0}\otimes\nu_n\right)$. The following step gives such a probability measure on $(\mathcal{K}_0,\delta^h)$ which pushforward measure by the function $\alpha_0(K)$ has truncated normal distribution on the range interval $[\frac{1}{2},1)$. We identified $\mathcal{K}_0$ with $\mathcal{K}^1_0\times[0,\infty)$, and introduced Φ_4 as the mapping $\Phi_4 : (K,\alpha)\mapsto \alpha K$. Finally we can identify $\mathcal{K}_0$ with $\mathcal{K}^1_0\times[0,\infty)$. To this end let Φ_4 be the mapping $\Phi_4 : (K,\alpha)\mapsto \alpha K$.

LEMMA 3.3.3. [10] *From the image $K' = \Phi_4(K)$ we can determine uniquely the body K and the constant α.*

PROOF. $K' = \alpha K$ implies that $\alpha_0(K) = \alpha_0(K') = \frac{d(K')}{w(K')+d(K')}$ and thus $\alpha_0(K)$ is uniquely determined. We also know the value of $\alpha' := \delta^h(\alpha K, B_E)$. We are considering two cases. In the first case we assume that $\alpha\geq\alpha_0$ and hence by Lemma 3.3.1 we get that $\alpha' = 2\alpha-1$ or $\alpha = \frac{\alpha'+1}{2}$, and in the second one we assume $0\leq\alpha\leq\alpha_0$ then we have $\alpha' = 2\alpha+1-2\frac{\alpha}{\alpha_0}$ or $\alpha = \frac{\alpha'-1}{2-\frac{2}{\alpha_0}} = \frac{\alpha_0(\alpha'-1)}{2(\alpha_0-1)}$. From these we get that the first case implies $\alpha_0\leq\frac{\alpha'+1}{2}$ so $\alpha'\geq 2\alpha_0-1$ and in the second one we have $\alpha_0\geq\frac{\alpha_0(\alpha'-1)}{2(\alpha_0-1)}\geq 0$. Hence we get $2\alpha_0-1\geq\alpha'\geq 0$. So first we determine α' and the value

$$2\alpha_0 - 1 = \frac{2\mathrm{diam}K}{\mathrm{w}(K)+\mathrm{diam}K} - 1 = \frac{\mathrm{diam}K - \mathrm{w}(K)}{\mathrm{diam}K+\mathrm{w}(K)}.$$

Then using the above equalities we can calculate α which is uniquely determined. Now K is equal to $\frac{1}{\alpha}K'$. □

Denote by $\Phi_4^{-1}(K') := \left(\left(\Phi_4^{-1}\right)_1(K'),\left(\Phi_4^{-1}\right)_2(K')\right)$ the pair (K,α) determined by the method of Lemma 3.3.3. If we have a σ-finite measure ν^1_0 on $\mathcal{K}^1_0$ then we also have a σ-finite measure ν_0 on $\mathcal{K}_0$ by the definition

$$\nu_0 = \Phi_4(\nu^1_0\otimes\nu),$$

where ν is a σ-finite measure on $(0,\infty)$.
Define the set function $p(\mathcal{A})$ as follows. If $\mathcal{A}\subset\mathcal{K}_0$ ν_0 is a measurable set let be

$$p(\mathcal{A}) := \frac{1}{\sqrt{2\pi\sigma^2}}\int\limits_{K'\in\mathcal{A}} e^{-\frac{\left(\delta^h\left(B_E,\frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2}K'\right)\right)^2}{2\sigma^2}}\mathrm{d}\nu_0.$$

The main result of [10] is:

THEOREM 3.3.2 ([10]). *If ν_0^1 is such a probability measure on $\mathcal{K}_0^1$ for which $\alpha_0(K)^{-1}(\nu_0^1)$ has uniform distribution, $\nu_0 = \Phi_4(\nu_0^1 \otimes \nu)$ where ν is a probability measure on $(0,\infty)$ and Φ is the probability function of the standard normal distribution then*

$$P(\mathcal{A}) := \frac{4p(\mathcal{A})}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)} = \frac{4}{\left(\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)\right)\sqrt{2\pi\sigma^2}} \int\limits_{K' \in \mathcal{A}} e^{-\frac{\left(\delta^h\left(B_E, \frac{\alpha_0(K')}{\Phi_4^{-1}(K')_2} K'\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu_0$$

is a probability measure on $\mathcal{K}_0$. Moreover $\alpha_0(K)^{-1}(P)$ has truncated normal distribution on the interval $[\frac{1}{2}, 1)$, (with mean $\frac{1}{2}$ and variance $\left(\frac{\sigma}{2}\right)^2$), so

$$\alpha_0(K)^{-1}(P)\left(\left\{\frac{1}{2} \leq t \leq c\right\}\right) = P\left(\{\mathcal{K} \in \mathcal{K}_0 \mid \alpha_0(K) \leq c\}\right) = \frac{\Phi\left(\frac{c-\frac{1}{2}}{\frac{\sigma}{2}}\right) - \Phi(0)}{\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)}.$$

PROOF.

$$p(\mathcal{A}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{K \in \left(\Phi_4^{-1}\right)_1(\mathcal{A})} \int\limits_{\alpha \in \left(\Phi_4^{-1}\right)_2(\mathcal{A})} e^{-\frac{\left(\delta^h\left(B_E, \frac{\alpha_0(K')}{\alpha} \alpha K\right)\right)^2}{2\sigma^2}} \mathrm{d}\nu \mathrm{d}\nu_0^1$$

however $\alpha_0(K') = \alpha_0(K)$ so it is equal to

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{K \in \left(\Phi_4^{-1}\right)_1(\mathcal{A})} \left(\int\limits_{\alpha \in \left(\Phi_4^{-1}\right)_2(\mathcal{A})} e^{-\frac{\alpha_0(K)'^2}{2\sigma^2}} \mathrm{d}\nu \right) \mathrm{d}\nu_0^1 =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{K \in \left(\Phi_4^{-1}\right)_1(\mathcal{A})} \left(\int\limits_{\substack{\alpha K \in \mathcal{A} \\ \alpha \geq \alpha_0(K)}} e^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu + \int\limits_{\substack{\alpha K \in \mathcal{A} \\ 0 \leq \alpha \leq \alpha_0(K)}} e^{-\frac{\left(2\alpha_0(K)+1-2\frac{\alpha_0(K)}{\alpha_0(K)}\right)^2}{2\sigma^2}} \mathrm{d}\nu \right) \mathrm{d}\nu_0^1 =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_{K \in \left(\Phi_4^{-1}\right)(\mathcal{A})_1} \left(\int\limits_{\alpha \in \left(\Phi_4^{-1}\right)_2(\mathcal{A})} e^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu \right) \mathrm{d}\nu_0^1 = \frac{\nu\left(\alpha \in \left(\Phi_4^{-1}\right)_2(\mathcal{A})\right)}{\sqrt{2\pi\sigma^2}} \int\limits_{K \in \left(\Phi_4^{-1}\right)(\mathcal{A})_1} e^{-\frac{(2\alpha_0(K)-1)^2}{2\sigma^2}} \mathrm{d}\nu_0^1.$$

For $\mathcal{A} = \mathcal{K}_0$ we have that it is equal to

$$\frac{\nu((0,\infty))}{\sqrt{2\pi\sigma^2}} \int\limits_{\frac{1}{2}}^{1} e^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{d}\left(\alpha_0(\mathrm{K})^{-1}(\nu_0^1)(\mathrm{t})\right).$$

Since ν is a probability measure on $(0,\infty)$ and $\alpha_0(K)^{-1}(\nu_0^1)$ has uniform distribution on $[\frac{1}{2}, 1)$ so we have that

$$p(\mathcal{K}_0) = \frac{1}{2\sqrt{2\pi}\frac{\sigma}{2}} \frac{1}{2} \left(\int\limits_{-\infty}^{1} e^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{dt} - \int\limits_{-\infty}^{\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{dt} \right) = \frac{\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)}{4},$$

where the function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{\left(-\frac{u^2}{2}\right)} \mathrm{du}$$

is the standard normal distribution function.
Analogously, for the set $\mathcal{K}_0(c) := \{K' \in \mathcal{K}_0 \mid \alpha_0(K') = \alpha_0(K) \leq c\}$ we have

$$p(\mathcal{K}_0(c)) = \frac{\nu((0,\infty))}{\sqrt{2\pi\sigma^2}} \int\limits_{\frac{1}{2}}^{c} e^{-\frac{1}{2}\left(\frac{t-\frac{1}{2}}{\frac{\sigma}{2}}\right)^2} \mathrm{d}\left(\alpha_0(\mathrm{K})^{-1}(\nu_0^1)(\mathrm{t})\right) = \frac{\Phi\left(\frac{c-\frac{1}{2}}{\frac{\sigma}{2}}\right) - \Phi(0)}{4}$$

thus the measure

$$P(\mathcal{A}) := \frac{4}{\Phi\left(\frac{1}{\sigma}\right) - \Phi(0)} p(\mathcal{A})$$

is such a probability measure on $\mathcal{K}_0$ which pushforward by the function $\alpha_0(K)$ has normal distribution. □

3.3.3. Extraction the measure to a geometric probability measure. The existence of a measure with similar properties on the space $\mathcal{K}$ of convex bodies follows easily. In fact, let denote by $m(K) := \frac{1}{2}(K + (-K))$ where the addition means the Minkowski sum of convex bodies. The mapping $m : \mathcal{K} \longrightarrow \mathcal{K}_0$ is a continuous function on $\mathcal{K}$ and thus it defines a pullback measure μ on $\mathcal{K}$ by the rule $\mu(H) = P(m(H))$ where $H = m^{-1}(H')$ for a Borel set $H' \in \mathcal{K}_0$. Observe that m has the following properties:

(1) surjective
(2) for any set $S \subset \mathcal{K}$ and a vector $t \in \mathbb{R}^n$ we have $m(S + t) = m(S)$
(3) for any $K \in \mathcal{K}$ holds that $\mathrm{diam}K = \mathrm{diam}(m(K))$, $\mathrm{w}(K) = \mathrm{w}(m(K)$ implying that $\alpha_0(K) = \alpha_0(m(K))$.

This implies that the function α_0 is well-defined on $\mathcal{K}$ and for any Borel set $B \in \left[\frac{1}{2}, 1\right)$ $\mu\left(\alpha_0^{-1}(B)\right) = P(m\left(\alpha_0^{-1}(B)\right)) = P(\alpha_0^{-1}|_{\mathcal{K}_0}(B))$ showing that the pushforward of the measure μ has truncated normal distribution on the interval $\left[\frac{1}{2}, 1\right)$.
Note that this measure is a geometric measure in the sense that invariant under rigid motions. The basic questions on such a measure are: "Do the convex polytopes have measure zero, do the smooth bodies have positive measure, or does a neighborhood always have positive measure?" The previous construction we can modify such that the improved one solves positively the above questions.

LEMMA 3.3.4 ([10]). *Denote by $\mathcal{P}_0$ the set of O-symmetric convex polytopes. Then we have $P(\mathcal{P}_0) = 0$.*

PROOF. Introduce the sets $\mathcal{P}_0^1$ and $\widetilde{\mathcal{P}_0^1}$ as we did in the case of the O-symmetric bodies $\mathcal{K}_0$. By definition we have $\widetilde{\mu_0^1}\left(\widetilde{\mathcal{K}_0^1} \setminus \widetilde{\mathcal{P}_0^1}\right) = 1$ showing that $\widetilde{\mu_0^1}\left(\widetilde{\mathcal{P}_0^1}\right) = 0$. Thus

$$\widetilde{\nu_0^1}\left(\widetilde{\mathcal{P}_0^1}\right) = \int\limits_{\widetilde{\mathcal{P}_0^1}} \mathrm{d}\widetilde{\nu_0^1} = \int\limits_{\widetilde{\mathcal{P}_0^1}} \frac{4}{(w+4)^2} \mathrm{d}\widetilde{\mu_0^1} = 0,$$

and so

$$\nu_0^1\left(\mathcal{P}_0^1\right) = \Phi_{2a}^1\left(\widetilde{\mathcal{P}_0^1} \otimes \nu_n\right)\left(\widetilde{\mathcal{P}_0^1}, SO(n)\right) = 0.$$

Finally, we have $\nu_0\left(\mathcal{P}_0\right) = \Phi_4\left(\nu_0^1 \otimes \nu\right)\left(\mathcal{P}_0^1, [0, \infty)\right) = 0$ implying $p\left(\mathcal{P}_0\right) = P\left(\mathcal{P}_0\right) = 0$ as we stated. □

We define the new system in two steps.

- Change the body $K_{m(n,k)}$ to a smooth body $K_{m(n,k)}^l$ defined by the convex hull of the ball around the origin with radius $m(n,k)$ and the two balls of radius $\varepsilon_l = \frac{1}{2^l} m(n,k)$ with centers $\pm(2 - \varepsilon_l)e_n$.
- Substitute each elements of the system of the bodies $K_{m(n,k)}^l$ with a new countable system of bodies. Consider a dense, countable and centrally symmetric point system $\{P_1, -P_1, P_2, -P_2 \cdots\}$ in the closed ball of radius 2 with the additional property that there is no two distances between the pairs of points which are equals to each other. (Such a point system is exist.) We assume that the first point P_1 is the endpoint of $2e_n$ and denote by S_i a similarity of E^n which sends P_1 into P_i and the ball of radius 2 at the origin into the ball of radius OP_i centered at the origin O, too. Consider the countable set of bodies $S\left(K_{m(n,k)}^l\right) := \left\{S_i\left(K_{m(n,k)}^l\right)\ , i = 1, 2, \ldots\right\}$ and define the elements of the new set $\mathcal{H}_{m(n,k)}^l$ by induction as follows:

– The first element is itself the set $K^l_{m(n,k)} := S_1\left(K^l_{m(n,k)}\right)$.
– In the second step consider such pairs from the list $S\left(K^l_{m(n,k)}\right)$ one of which has diameter 4 and construct their convex hulls. Add these bodies also to the set $\mathcal{H}^l_{m(n,k)}$.
– In the third step construct the convex hull of the triplet from which one has diameter 4. Add these bodies to $\mathcal{H}^l_{m(n,k)}$, too.
– ... and so on.

Hence we have a countable system of centrally symmetric convex bodies with diameter 4. The getting set $\mathcal{H}^l_{m(n,k)}$ has a partition into countable subsets. So we have:

$$\mathcal{H}^l_{m(n,k)} = K^l_{m(n,k)} \dot\cup \left\{\text{conv}\left\{S_i\left(K^l_{m(n,k)}\right), S_j\left(K^l_{m(n,k)}\right)\right\} \text{ for } i,j\right\} \dot\cup$$
$$\dot\cup \left\{\text{conv}\left\{S_i\left(K^l_{m(n,k)}\right), S_j\left(K^l_{m(n,k)}\right), S_k\left(K^l_{m(n,k)}\right)\right\} \text{ for } i,j,k\right\} \dot\cup \cdots,$$

where all of the elements are smooth bodies having diameter 4. The following technical lemma is important.

LEMMA 3.3.5 ([10]). *The bodies of*

$$\mathfrak{H} = \left\{\mathcal{H}^l_{m(n,k)} \quad m,n,k,l \in \mathbb{N}\right\}$$

are pairwise non-congruent. For an arbitrary polytope $Q \in P_0$ and for a given number ε we can choose an element $R \in \mathfrak{H}$ for which hold that $\delta^h(Q,R) < \varepsilon$.

PROOF. The first statement follows from the fact that each of the bodies of $\mathfrak{H}$ contains a maximal flat part which is the convex hulls of the points P_i. By the choice of the point system $\{P_i\}$ these parts are pairwise non-congruent. The proof of the second statement based on the fact that for large l, $m(n,k)$ with a small k the bodies $S\left(K^l_{m(n,k)}\right)$ essentially are O-symmetric segments and thus their convex hull is close to a polytope with respect to the Hausdorff distance. We here omit the straightforward argument. □

We are distributing among the elements of $\mathcal{H}^l_{m(n,k)}$ that part of the measure $\widetilde{\mu^1_0}$ which originally concentrated on $K^l_{m(n,k)}$.
For a fixed $r \in \mathbb{N}$ consider a sequence (α^r_i) of positive numbers which holds the property $\sum\limits_{i=1}^{\infty} \alpha^r_i = 1$. Let $L^r_i(l)$ be the i^{th} element of the r-th subset of the above partition of $\mathcal{H}^l_{m(n,k)}$. Thus it is a convex hull of exactly r copies of bodies from $S\left(K^l_{m(n,k)}\right)$. We give it the weight $\alpha^r_i/2^r$.

DEFINITION 3.3.2 ([10]). *Choose a sequence of positive numbers β_l with again the property $\sum\limits_{l=1}^{\infty} \beta_l = 1$. Define a measure $\widetilde{\widetilde{\mu^1_0}}$ by the equality:*

$$\widetilde{\widetilde{\mu^1_0}} := \lim_{n\to\infty} \sum_{k=1}^{2^{n+1}} \sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} \frac{\beta_l \alpha^r_i}{2^{n+1+r}} \delta_{L^r_i(l)}.$$

We prove the following theorem:

THEOREM 3.3.3 ([10]). *On the space of norms there is a probability measure P with the following properties:*

- *The neighborhoods has positive measure.*
- *The set of polytopes has zero measure.*
- *The set of smooth bodies has measure* 1.
- *The pushforward $\alpha_0(K)^{-1}(P)$ of P has truncated normal distribution on the interval $[\frac{1}{2}, 1)$.*

PROOF. Consider the measure $\widetilde{\widetilde{\mu}}_0^1$ without the measure $\widetilde{\mu}_0^1$ and expand it for $\mathcal{K}_0$ on the way as we did it with $\widetilde{\mu}_0^1$. The final measure P by Lemma 3.3.4 on the set of polytopes has zero value. By the remark before the definition of the new system we know that the set of smooth bodies of $\mathcal{K}_0$ has measure 1 since the elements of $\mathfrak{H}$ are smooth. The required property on the approximation of polytopes follows from Lemma 3.3.5 since for each polytope we can find a body from $\mathcal{H}^l_{m(n,k)}$ close to them. The definition of $\widetilde{\widetilde{\mu}}_0^1$ guarantees that the distribution of $\widetilde{\widetilde{\mu}}_0^1$ and $\widetilde{\mu}_0^1$ are agree proving our last statement. □

3.4. Generalized space-time model with changing shape

Our investigation on space-time originated from Minkowski, Lorentz, Einstein and Riemann. Minkowski observed (see [**123**]) that the mathematical structure of special relativity requires a special kind of geometry the geometry of space-time. In space-time we have a homogeneous system of points in each point we can measure the distance at the same manner. Locally we have only three types of points which are agree one of the space-like, time-like and light-like properties, respectively. Global relativity rewrote this concept, the existence of gravity changes the geometric structure of the space hence we cannot consider our world such a manifold which has the same local metric geometry in its points independently from the position of the points and the date of the event. In such a model the metric of the geometry changes by point to point. The description in its full generalization require the Riemann geometric approach in which the leading role of the time is loose. To approach global relativity theory we should use the mathematical background of a Lorentzian manifold in which the points of the world don't ordered by the time. Though this generalization is necessary for a complete handing of this problem there are many important situation in which the ordering role of the time natural and indisputable. Our goal is to create an immediate structure between space-time and Lorentzian manifold suitable to describe those phenomenon in which the time has an important role. For this purpose we give in [**11**] a mathematical model called by time-space in two versions (one of them deterministic and the other one is random) and prove that substantially all of them can be considered relevant. The knowledge of the author either this model and the corresponding investigations are new. On the other hand there is fully developed theory which can be followed in this situation. Hence the results in this paper can be valued differently. We concentrate in this thesis only such things which fully understandable for a pure mathematics.

3.4.1. Deterministic time-space model. We assume that there is an absolute coordinate system of dimension n in which we are modeling the universe by a time-space model. The origin is a generalized space-time model in which the time axis plays the role of the absolute time. In a fixed moment (with respect to this absolute time) the collection of the points of the space can be regarded as an open punctured ball of the embedding normed space which is centered at the origin that does not contain the origin. The omitted point is the origin of a coordinate system giving the space-like coordinates of the world-points with respect to our time-space system. Since the points of the axis of the absolute-time are not in our universe there is no reference system in our modeled world which determines the absolute time.[2]
In our deterministic model (based on a generalized space-time model) the absolute coordinates of points are calculated by a fixed basis of the embedding vector space. The vector $s(\tau)$ means the collection of the space-components with respect to the absolute time τ, the quantity τ has to be measured on a line T which orthogonal to the linear subspace S of the vectors $s(\tau)$. (The orthogonality was considered as the Pythagorean orthogonality of the embedding normed space.) Consider a fixed Euclidean vector space with unit ball B_E on S and use its usual

[2]In mathematical point of view there is no importance that the absolute time-axis can be found (is "exists") or cannot be found (is not "exists"). In our calculations assume that the shape of the universe in a moment is an open centrally symmetric convex body. Its center is also unknown and we can visualize it as a point of the axis of absolute-time.

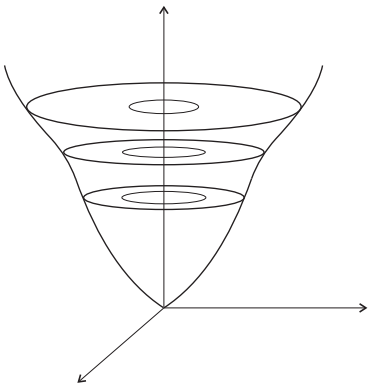

FIGURE 3.3. The shape of the universe.

functions e.g. volume, diameter, width, thinness and Hausdorff distance. With respect to the moment τ of the absolute time we have a unit ball $K(\tau)$ in the corresponding normed space $\{S, \|\cdot\|^\tau\}$. The modeled universe at τ is the ball $\tau K(\tau) \subset \{S, \|\cdot\|^\tau\}$. The shape of the model at the moment τ depends on the shape of the centrally symmetric convex body $K(\tau)$. The center of the model is on the axis of the absolute time, it cannot be determined. For calculations on time-space we need further smoothness properties on $K(\tau)$. These are

- $K(\tau)$ is a centrally symmetric, convex, compact, C^2 body of volume $\mathrm{vol}(B_E)$.
- For each pairs of points s', s'' the function $K : \mathbb{R}^+ \cup \{0\} \to \mathcal{K}_0$, $\tau \mapsto K(\tau)$ holds the property that $[s', s'']^\tau : \tau \mapsto [s', s'']^\tau$ is a C^1-function.

DEFINITION 3.4.1. [11] *We say that a generalized space-time model endowed with a function $K(\tau)$ holding the above properties is a* deterministic time-space model.

The main subset of a deterministic time-space model contains the points of negative norm-square. This is the set of time-like points and the upper connected sheet of the time-like points is the modeled universe. The points of the universe have positive time-components. We denote this model by $(M, K(\tau))$.
We remark that in the two-dimensional case for each τ, $K(\tau)$ is a segment with length two, thus our model is the 2-dimensional space-time. On the other hand, with n greater than or equal to 3, the two-dimensional space-time sections of our general space-time bounded by general (non-convex) curves symmetric about the time-axis (see on Fig. 3.3).
We can give a product similar to the Minkowski product of a generalized space-time model. In a two-dimensional plane the role of the light-cone play the curve $[\alpha^e(\tau)e, \alpha^e(\tau)e]^\tau + [\tau, \tau] = 0$. For a fixed direction x, we consider the curves $t_{\beta,e} : \tau \mapsto \beta\alpha^e(\tau)e + \tau e_n$ through the point $x = \beta\alpha^e(\tau)e + \tau e_n$. Note that x is a time-like point if $|\beta| < 1$. The role of the imaginary unit sphere is played by the set of points

$$\cup \left\{ \left\{ s + \tau \text{ where } \sqrt{[s,s]^\tau + 1} = \tau \right\} \ , \tau \geq 1 \right\}.$$

In the direction of e it is a curve defined by the implicit equation $\sqrt{[s,s]^\tau + 1} = \tau$, $\tau \geq 1$. The intersection of this curve with $t_{\beta,e}$ is a point satisfying the equality $[\beta\alpha^e(\tau^\star)e, \beta\alpha^e(\tau^\star)e]^{\tau^\star} + 1 = (\tau^\star)^2$, with parameter $\tau^\star$, and hence we get $\beta^2 (\tau^\star)^2 + 1 = (\tau^\star)^2$, or equivalently $(\tau^\star)^2 = \frac{1}{1-\beta^2}$. Assuming that our examination is on the positive part of the set of time-like points we have $\tau^\star = \frac{1}{\sqrt{1-\beta^2}}$ or $\beta = \frac{\sqrt{(\tau^\star)^2-1}}{\tau^\star}$.
In the space-time model the tangent of the imaginary unit curve is orthogonal to the position vector of the common point. This requires that in the case of generalized space-time model, the product

$$\left[e + \left(\sqrt{[s,s]^\tau + 1} \right)'_e (\beta\alpha^e(\tau^\star)e)\, e_n, \beta\alpha^e(\tau^\star)\, e + \tau^\star e_n \right]$$

will be equal to zero. Another claim that the product is equal to the corresponding norm-square in the case when its arguments contains the same vectors. We will need a lemma on the directional derivative of the function which defines the imaginary unit sphere.

LEMMA 3.4.1 ([11]). *The directional derivative of the real valued function* $\mathfrak{h}(s) = \sqrt{[s,s]^{\mathfrak{h}(s)} + 1}$ *is*

$$\mathfrak{h}'_e(s) = \left(1 - \frac{\frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s))}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}\right)^{-1} \frac{[e,s]^{\mathfrak{h}(s)}}{\sqrt{1+[s,s]^{\mathfrak{h}(s)}}} = \frac{2}{2\mathfrak{h}(s) - \frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s))}[e,s]^{\mathfrak{h}(s)},$$

or equivalently

$$\mathfrak{h}'_e(s) = \frac{1}{\mathfrak{h}(s) - \|s\|^{\mathfrak{h}(s)} \frac{\partial \|s\|^\tau}{\partial \tau}(\mathfrak{h}(s))}[e,s]^{\mathfrak{h}(s)}.$$

PROOF. The considered derivative is

$$\mathfrak{h}'_e(s) = \frac{1}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}([s,s]^{\mathfrak{h}(s)})'_e.$$

It can be seen easily (or use the calculation of Theorem 3.4.1 with the substitutions $c(t+\lambda) = s + \lambda e$, $(f_1)_S = (f_2)_S = \mathrm{id}|_S$ and $(f_1)_T = (f_2)_T = \mathfrak{h}$) that the directional derivative is equal to

$$\mathfrak{h}'_e(s) = \frac{1}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}\left([e,s]^{\mathfrak{h}(s)} + \left([s,\cdot]^{\mathfrak{h}(s)}\right)'_e(s) + \frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s)) \cdot (\mathfrak{h})'_e(s)\right) =$$

$$= \frac{1}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}\left(2[e,s]^{\mathfrak{h}(s)} + \frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s)) \cdot (\mathfrak{h})'_e(s)\right).$$

Thus we get

$$\mathfrak{h}'_e(s)\left(1 - \frac{\frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s))}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}\right) = \frac{[e,s]^{\mathfrak{h}(s)}}{\sqrt{1+[s,s]^{\mathfrak{h}(s)}}},$$

or equivalently the required formulas

$$\mathfrak{h}'_e(s) = \left(1 - \frac{\frac{\partial [s,s]^\tau}{\partial \tau}(\mathfrak{h}(s))}{2\sqrt{1+[s,s]^{\mathfrak{h}(s)}}}\right)^{-1} \frac{[e,s]^{\mathfrak{h}(s)}}{\sqrt{1+[s,s]^{\mathfrak{h}(s)}}} = \frac{1}{\mathfrak{h}(s) - \|s\|^{\mathfrak{h}(s)} \frac{\partial \|s\|^\tau}{\partial \tau}(\mathfrak{h}(s))}[e,s]^{\mathfrak{h}(s)}.$$

□

DEFINITION 3.4.2 ([11]). *For two vectors* $s_1 + \tau_1$ *and* $s_2 + \tau_2$ *of the deterministic time-space model define their product with the equality*

$$[s_1 + \tau_1, s_2 + \tau_2]^{+,T} := [s_1, s_2]^{\tau_2} + [\tau_1, \tau_2] = [s_1, s_2]^{\tau_2} - \tau_1\tau_2.$$

Here $[s_1, s_2]^{\tau_2}$ means the s.i.p defined by the norm $\|\cdot\|^{\tau_2}$. This product is not a Minkowski product, as there is no homogeneity property in the second variable. On the other hand the additivity and homogeneity properties of the first variable, the properties on non-degeneracy of the product are again hold, and the continuity and differentiability properties of this product also remain the same as of a Minkowski product. The calculations in a generalized space-time model basically depend on a rule on the differentiability of the second variable of the Minkowski product. Using the notation

$$[f_1(c(t)),\cdot]^{+\prime}_{D(f_2\circ c)(t)}(f_2(c(t))) := \left([(f_1)_S(c(t)),\cdot]'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t))) - (f_1)_T(c(t))((f_2)_T \circ c)'(t)\right),$$

in Lemma 3.2.4 we stated that if $f_1, f_2 : S \longrightarrow V = S + T$ are two C^2 maps and $c : \mathbb{R} \longrightarrow S$ is an arbitrary C^2 curve then

$$([(f_1 \circ c)(t)), (f_2 \circ c)(t))]^+)' = [D(f_1 \circ c)(t), (f_2 \circ c)(t))]^+ + [(f_1 \circ c)(t)),\cdot]^{+\prime}_{D(f_2\circ c)(t)}((f_2 \circ c)(t)).$$

Regarding to the importance of this rule we reproduce it in a time-space model. Let denote by f_S and f_T the component functions of f with respect to the subspaces S and T, respectively. By definition, let us denote

$$\left([f_1(c(t)),\cdot]^{+,T}\right)'_{D(f_2\circ c)(t)}(f_2(c(t))) :=$$

$$= \left([(f_1)_S(c(t)),\cdot]^{(f_2)_T(c(t))}\right)'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t))) - (f_1)_T(c(t))((f_2)_T\circ c)'(t) +$$
$$+(f_1)_T(c(t))\frac{\partial^2[(f_2)_S(c(t)),(f_2)_S(c(t))]^\tau}{(\partial\tau)^2}((f_2)_T(c(t)))\,[D((f_2)_S\circ c)(t),(f_2)_S(c(t))]^{(f_2)_T(c(t))}.$$

We now generalize the formula of Lemma 3.2.4.

THEOREM 3.4.1 ([11]). *If $f_1, f_2 : S \longrightarrow V = S + T$ are two C^2 maps and $c : \mathbb{R} \longrightarrow S$ is an arbitrary C^2 curve then*

$$([(f_1\circ c)(t)),(f_2\circ c)(t))]^{+,T})' = [D(f_1\circ c)(t), f_2(c(t))]^{+,T} + \left([f_1(c(t)),\cdot]^{+,T}\right)'_{D(f_2\circ c)(t)}(f_2(c(t))) +$$
$$+\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))]^\tau}{\partial\tau}((f_2)_T(c(t)))\cdot((f_2)_T\circ c)'(t)$$

PROOF. By definition

$$([f_1\circ c, f_2\circ c)]^{+,T})'|_t := \lim_{\lambda\to 0}\frac{1}{\lambda}\left([f_1(c(t+\lambda)), f_2(c(t+\lambda))]^{+,T} - [f_1(c(t)), f_2(c(t))]^{+,T}\right) =$$
$$= \lim_{\lambda\to 0}\frac{1}{\lambda}\left([(f_1)_S(c(t+\lambda)),(f_2)_S(c(t+\lambda))]^{(f_2)_T(c(t+\lambda))} - [(f_1)_S(c(t)),(f_2)_S(c(t))]^{(f_2)_T(c(t))}\right) +$$
$$+\lim_{\lambda\to 0}\frac{1}{\lambda}\left([(f_1)_T(c(t+\lambda)),(f_2)_T(c(t+\lambda))] - [(f_1)_T(c(t)),(f_2)_T(c(t))]\right).$$

The first part can be written in the form

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\left([(f_1)_S(c(t+\lambda)) - (f_1)_S(c(t)),(f_2)_S(c(t+\lambda))]^{(f_2)_T(c(t+\lambda))} +\right.$$
$$\left.+[(f_1)_S(c(t)),(f_2)_S(c(t+\lambda))]^{(f_2)_T(c(t+\lambda))} - [(f_1)_S(c(t)),(f_2)_S(c(t))]^{(f_2)_T(c(t))}\right).$$

We prove that it is equal to

$$[D((f_1)_S\circ c)|_t,(f_2)_S(c(t))]^{(f_2)_T(c(t))} + \left([(f_1)_S(c(t)),\cdot]^{(f_2)_T(c(t))}\right)'_{D((f_2)_S\circ c)(t)}((f_2)_S(c(t))) +$$
$$+\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))]^\tau}{\partial\tau}((f_2)_T(c(t)))\cdot((f_2)_T\circ c)'(t).$$

In this latter equation the first term comes from the value of the first bracket of the earlier one. We calculate now the remaining substraction. For this, take the fixed (absolute) coordinate system $\{e_1,\cdots,e_{n-1}\}$ of S and consider the coordinate-wise representation $(f_2)_S\circ c = \sum_{i=1}^{n-1}((f_2)_S\circ c)_i e_i$. Using Taylor's theorem for the coordinate functions we have that there are real parameters $t_i \in (t, t+\lambda)$, for which

$$((f_2)_S\circ c)(t+\lambda) = ((f_2)_S\circ c)(t) + \lambda D((f_2)_S\circ c)(t) + \frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)_i''(t_i)e_i.$$

Thus we get that

$$[(f_1)_S(c(t)),(f_2)_S(c(t+\lambda))]^{(f_2)_T(c(t+\lambda))} - [(f_1)_S(c(t)),(f_2)_S(c(t))]^{(f_2)_T(c(t))} =$$
$$= \left[(f_1)_S(c(t)),(f_2)_S(c(t)) + D((f_2)_S\circ c)(t)\lambda + \frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)_i''(t_i)e_i\right]^{(f_2)_T(c(t+\lambda))} -$$
$$- [(f_1)_S(c(t)),(f_2)_S(c(t))]^{(f_2)_T(c(t))} =$$
$$= \left([(f_1)_S(c(t)),(f_2)_S(c(t)) + D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t))} - [(f_1)_S(c(t)),(f_2)_S(c(t))]^{(f_2)_T(c(t))}\right) +$$
$$+\left(\left[(f_1)_S(c(t)),(f_2)_S(c(t)) + D((f_2)_S\circ c)(t)\lambda + \frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)_i''(t_i)e_i\right]^{(f_2)_T(c(t+\lambda))} -\right.$$
$$\left.- [(f_1)_S(c(t)),(f_2)_S(c(t)) + D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t))}\right).$$

Dividing by λ and applying the limit procedure when λ tends to zero we get from the first bracket the value:

$$\left([(f_1)_S(c(t)),\cdot]^{(f_2)_T(c(t))}\right)'_{D((f_2)_S\circ c)(t)}(((f_2)_S\circ c)(t))).$$

We also determine the value of the second bracket. By Definition 3.4.2 the second term in this bracket is

$$[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t))}=[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t+\lambda))}-$$
$$-\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t))}}{\partial\tau}\lambda'-o(\lambda'),$$

where $(f_2)_T(c(t+\lambda))=(f_2)_T(c(t))+\lambda'$ and $\lim\limits_{\lambda'\mapsto 0}\frac{o(\lambda')}{\lambda'}=0$.

Since $(f_2)_Tc(t+\lambda)=(f_2)_Tc(t)+\lambda\left((f_2)_T\circ c\right)'(t)+o_1(\lambda)$, we have that $\lambda'=\lambda\left((f_2)_T\circ c\right)'(t)+o_1(\lambda)$. By the Lipschitz condition (which also holds in the second variable of the product) there is a real constant K with which we have that the absolute value of the substraction

$$\left[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda+\frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)''_i(t_i)e_i\right]^{(f_2)_T(c(t+\lambda))}-$$
$$-[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t+\lambda))}$$

is less than or equal to

$$K\left[(f_1)_S(c(t)),\frac{1}{2}\lambda^2\sum_{i=1}^{n-1}((f_2)_S\circ c)''_i(t_i)e_i\right]^{(f_2)_T(c(t+\lambda))}.$$

Dividing by λ and applying the limit procedure as $\lambda\to 0$, this quantity tends to zero. Dividing also by λ, for the remaining parts we have

$$\frac{1}{\lambda}\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))+D((f_2)_S\circ c)(t)\lambda]^{(f_2)_T(c(t))}}{\partial\tau}\lambda'+o(\lambda')=$$
$$=\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))+\lambda D((f_2)_S\circ c)(t)]^{(f_2)_T(c(t))}}{\partial\tau}\left(\left((f_2)_T\circ c\right)'(t)+\frac{o_1(\lambda)}{\lambda}\right)+$$
$$+\left(\frac{o\left(\lambda\left((f_2)_T\circ c\right)'(t)+o_1(\lambda)\right)}{\lambda\left((f_2)_T\circ c\right)'(t)+o_1(\lambda)}\right)\left(\frac{\lambda\left((f_2)_T\circ c\right)'(t)+o_1(\lambda)}{\lambda}\right),$$

and if λ tends to zero then it is equal to

$$\frac{\partial[(f_1)_S(c(t)),(f_2)_S(c(t))]^{\tau}}{\partial\tau}((f_2)_T(c(t)))\cdot((f_2)_T\circ c)'(t).$$

Thus, we proved our statement on the space-like component. On the other hand $(f_1)_T$, $(f_2)_T$, are real-real functions, respectively. This implies that

$$\lim_{\lambda\to 0}\frac{1}{\lambda}([(f_1)_T(c(t+\lambda)),(f_2)_T(c(t+\lambda))]-[(f_1)_T(c(t)),(f_2)_T(c(t))])=$$
$$=-((f_1)_T\circ c)'(t)(f_2)_T(c(t))-(f_1)_T(c(t))((f_2)_T\circ c)'(t)$$

showing the assertion of the theorem. □

In a deterministic time-space model we should investigate such $n-1$-dimensional subsets which cannot be considered globally as a hypersurface but locally holds this property.

3.4.1.1. *Imaginary unit sphere of a deterministic time-space model.* The points of $H^{+,T}$ can be defined by the union $\cup\left\{\left\{s+\tau \text{ where } \sqrt{[s,s]^{\tau}+1}=\tau\right\}\ ,\ \tau\geq 1\right\}$. Our assumption on $K(\tau)$ cannot guaranties that for every $s\in S$ there is a τ which holds the equality $\sqrt{[s,s]^{\tau}+1}=\tau$. On the other hand if we assume that $\rho_H(K(\tau),B_E)\leq 1$ the ball $2K(\tau)$ contains the Euclidean ball B_E for every τ. Hence $[s,s]^{\tau}\leq 4\|s\|_E^2$ so for all τ with $\tau^2>4\|s\|_E^2+1$, the inequality $[s,s]^{\tau}+1<\tau^2$ holds. Since for a non-zero vector s we have also that $[s,s]^1+1>1$, the statement follows by continuity. Clearly, $H^{+,T}$ generally cannot be considered as a hypersurface of the time-space implying that its differential geometry can be considered only on the base of its implicit definition. On the other hand we can consider the function $\mathfrak{H}:V\to\mathbb{R}$ defined by $\mathfrak{H}(s+\tau e_n):=\sqrt{[s,s]^{\tau}+1}-\tau$. If $v_0=s_0+\tau_0 e_n$ is a point on $H^{+,T}$ then we have $\mathfrak{H}(v_0)=0$. By our definition $\mathfrak{H}$ is continuously differentiable at the point v_0. Assume that

$$\frac{\partial\mathfrak{H}}{\partial\tau}(v_0)=\frac{\frac{\partial([s,s]^{\tau})}{\partial\tau}}{2\sqrt{[s,s]^{\tau}+1}}(v_0)-1\neq 0,\ \text{or equivalently}\ \frac{\partial([s_0,s_0]^{\tau})}{\partial\tau}(\tau_0)\neq 2\sqrt{[s_0,s_0]^{\tau_0}+1}.$$

Then by the implicit function theorem there is a neighborhood U of v_0 and a function $\mathfrak{h}:S\to\mathbb{R}$ such that $\tau=\mathfrak{h}(s)$ hold for the points $v=s+\tau e_n$ of $H^{+,T}$. Thus we have in U (as in Lemma 3.4.1) that $\mathfrak{h}(s)=\sqrt{[s,s]^{\mathfrak{h}(s)}+1}$. If the vector s comes from a point of a curve $c(t)\subset S$ by the definition $c(t+\lambda):\lambda\to s+\lambda e$, we get the equality: $(\mathfrak{h}\circ c)(t)=\sqrt{[(c(t),c(t)]^{\mathfrak{h}(c(t))}+1}$ and also

$$(\mathfrak{h}\circ c)'(t)=\frac{[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}+\frac{\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))\cdot(\mathfrak{h}\circ c)'(t)}{2\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}$$

or equivalently,

$$(\mathfrak{h}\circ c)'(t)=\left(1-\frac{\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}{2\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}\right)^{-1}\frac{[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}=$$

$$=\frac{2}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}.$$

We note that the additional value

$$\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))$$

of the formula depend on the position $c(t+0)=s$ and do not depend on the direction vector e. Thus the first fundamental form is:

$$\mathrm{I}=[\dot{c}(t)+(\mathfrak{h}\circ c)'(t)e_n,\dot{c}(t)+(\mathfrak{h}\circ c)'(t)e_n]^{+,T}=[\dot{c}(t),\dot{c}(t)]^{(\mathfrak{h}\circ c)'(t)}-[(\mathfrak{h}\circ c)'(t)]^2=$$

$$=[\dot{c},\dot{c}]^{\frac{2[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}}-\left(\frac{2[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}\right)^2.$$

To calculate the second fundamental form we have to determine the unit normal vector field. A tangent vector is

$$\dot{c}(t)+(\mathfrak{h}\circ c)'(t)e_n=\dot{c}(t)+\left(1-\frac{\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}{2\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}\right)^{-1}\frac{[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{\sqrt{1+[c(t),c(t)]^{\mathfrak{h}(c(t))}}}e_n.$$

We can see that

$$\left[\dot{c}(t)+\frac{2[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}e_n,\frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}c(t)+\mathfrak{h}(c(t))e_n\right]^{+,T}=0$$

showing the equality

$$n\circ c=\frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t))-\frac{\partial[c(t),c(t)]^{\tau}}{\partial\tau}(\mathfrak{h}(c(t)))}c(t)+(\mathfrak{h}\circ c)(t)e_n.$$

The second fundamental form of $H^{+,T}$ is

$$\mathrm{II} := \left[\ddot{c}(t) + (\mathfrak{h}\circ c)''(t)e_n, \frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}c(t) + (\mathfrak{h}\circ c)(t)e_n\right]_{(\mathfrak{h}\circ c)(t)}^{+,T} =$$

$$= \frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}[\ddot{c}(t), c(t)]^{(\mathfrak{h}\circ c)(t)} - (\mathfrak{h}\circ c)''(t)\mathfrak{h}(c(t)).$$

In fact we can use here Theorem 3.4.1. Thus we get first that

$$(\mathfrak{h}\circ c)''(t) = \left(\frac{2\,[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\right)' = A + B$$

where

$$A = \frac{2}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\left([\ddot{c}(t), c(t)]^{\mathfrak{h}(c(t))} + \left([\dot{c}(t),\cdot]^{\mathfrak{h}(c(t))}\right)'_{\dot{c}(t)}(c(t)) + \right.$$

$$\left. + \frac{2[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))}\frac{\partial[\dot{c}(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\right)$$

and

$$B = \frac{-2[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))}}{\left(2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right)^2}2\left((\mathfrak{h}\circ c)'(t)\left(1 - \frac{1}{2}\frac{\partial^2[c(t),c(t)]^\tau}{(\partial\tau)^2}(\mathfrak{h}(c(t)))\right) - \right.$$

$$\left. - \frac{\partial[\dot{c}(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right) = \frac{-2}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\left(((\mathfrak{h}\circ c)'(t))^2\left(1 - \frac{1}{2}\frac{\partial^2[c(t),c(t)]^\tau}{(\partial\tau)^2}(\mathfrak{h}(c(t)))\right) - \right.$$

$$\left. - (\mathfrak{h}\circ c)'(t)\frac{\partial[\dot{c}(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right).$$

Since in time-space model we have $\left([\dot{c}(t),\cdot]^{\mathfrak{h}(c(t))}\right)'_{\dot{c}(t)}(c(t)) = [\dot{c}(t),\dot{c}(t)]^{(\mathfrak{h}\circ c)'(t))}$ we get that the second fundamental form is:

$$\mathrm{II} = \frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}[\ddot{c}(t), c(t)]^{\mathfrak{h}(c(t))} - (\mathfrak{h}\circ c)''(t)\mathfrak{h}(c(t)) =$$

$$= \frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\left[-[\dot{c}(t),\dot{c}(t)]^{(\mathfrak{h}\circ c)'(t)} + \right.$$

$$\left. + ((\mathfrak{h}\circ c)'(t))^2\left(1 - \frac{1}{2}\frac{\partial^2[c(t),c(t)]^\tau}{(\partial\tau)^2}(\mathfrak{h}(c(t)))\right) - 2(\mathfrak{h}\circ c)'(t)\frac{\partial[\dot{c}(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right],$$

where

$$(\mathfrak{h}\circ c)'(t)) = \frac{2[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}.$$

Observe, that if the norm is a constant function of the time, these formulas simplify to the formulas of the generalized space-time model. We now give three examples to illustrate that these important formulas can be calculated, concretely.

EXAMPLE 3.4.1. [11]
(1) For a 3-dimensional example we take the function $K(\tau) : \tau \mapsto G_\tau$, where G_τ is the ellipse of area π with half-axes τe_1 and $\frac{1}{\tau}e_2$. Here $\{e_1, e_2\}$ is an orthonormed basis of the embedding Euclidean plane. The connection between the norms of the vector $s = xe_1 + ye_2$ and its Euclidean coordinates is $[s, s]^\tau = \tau^2x^2 + \frac{y^2}{\tau^2}$. The implicite equation for the corresponding imaginary unit sphere is $\tau = \sqrt{1 + \tau^2x^2 + \frac{y^2}{\tau^2}}$, if we assume that $2\tau x^2 - \frac{2y^2}{\tau^3} \neq 2\tau$, or equivalently $x^2 - 1 \neq \frac{y^2}{\tau^4}$. For a vector $s = (x, y)^T$ we exclude the moment τ holding the equality $\tau^4 = \frac{y^2}{x^2-1}$ where $x^2 \neq 1$.

(Thus if $x^2 = 1$ there is no τ, which we should exclude from the investigation.) Solving the implicit equation we get that

$$\tau^2 = \frac{1 \pm \sqrt{1 + 4(1-x^2)y^2}}{2(1-x^2)} \text{ if } x^2 \neq 1,$$

and in the case when $x^2 = 1$ τ has to be ∞ for every y. This formula shows that we can get real values for τ if and only if $x^2 \leq 1 + \frac{1}{4y^2}$. Thus the domain of the imaginary unit sphere is the union of three domains bounded by the curves $x = \pm 1$ and $x = \pm\sqrt{1 + \frac{1}{4y^2}}$ drawing on the figure Fig 3.4.

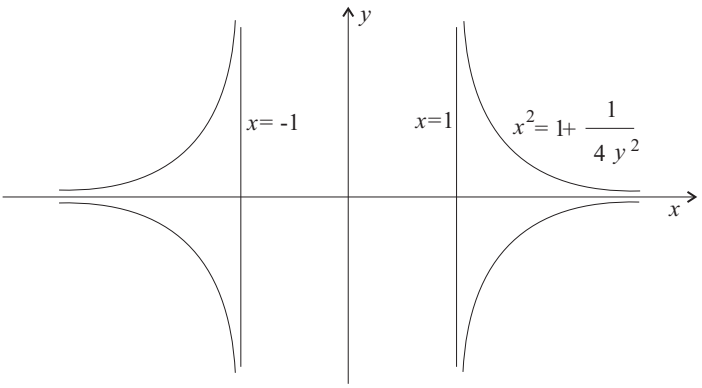

FIGURE 3.4. The domain of the imaginary unit sphere in the example.

Since $\tau^2 > 0$ we also have that if $|x| < 1$ then we have to consider the equality with positive sign

$$\tau^2 = \frac{1 + \sqrt{1 + 4(1-x^2)y^2}}{2(1-x^2)},$$

and for the other two connected components we have to choose the equality with negative sign:

$$\tau^2 = \frac{1 - \sqrt{1 + 4(1-x^2)y^2}}{2(1-x^2)}.$$

The first fundamental form is

$$\mathrm{I} = [\dot{c}, \dot{c}]^{\frac{2[\dot{c}(t),c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t)) - \frac{\partial [c(t),c(t)]^{\tau}}{\partial \tau}(\mathfrak{h}(c(t)))}} - \left(\frac{2[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))}}{2\mathfrak{h}(c(t)) - \frac{\partial [c(t),c(t)]^{\tau}}{\partial \tau}(\mathfrak{h}(c(t)))} \right)^2.$$

Since

$$[\dot{c}(t), c(t)]^{\mathfrak{h}(c(t))} = \mathfrak{h}(c(t))^2 \dot{x}(t) x(t) + \frac{\dot{y}(t) y(t)}{\mathfrak{h}(c(t))^2},$$

$$\frac{\partial [c(t), c(t)]^{\tau}}{\partial \tau}(\mathfrak{h}(c(t))) = 2\mathfrak{h}(c(t)) x(t)^2 - \frac{2y(t)^2}{\mathfrak{h}(c(t))^3},$$

we have that

$$\mathrm{I} = ((\mathfrak{h} \circ c)'(t))^2 \left((\dot{x}(t))^2 - 1 \right) + \frac{(\dot{y}(t))^2}{((\mathfrak{h} \circ c)'(t))^2},$$

where

$$(\mathfrak{h} \circ c)'(t) = \mathfrak{h}(c(t) \frac{(\mathfrak{h}(c(t))^4 \dot{x}(t) x(t) + \dot{y}(t) y(t)}{(\mathfrak{h}(c(t))^4 (1 - (x(t))^2) + (y(t))^2}$$

with

$$(\mathfrak{h}(c(t)))^2 = \frac{1 \pm \sqrt{1 + 4(1 - (x(t))^2)(y(t))^2}}{2(1 - (x(t))^2)}.$$

We also get that

$$\mathrm{II} = -\frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\left[((\mathfrak{h}\circ c)'(t))^2(\dot{x}(t))^2 + \right.$$
$$+\frac{(\dot{y}(t))^2}{((\mathfrak{h}\circ c)'(t))^2} - ((\mathfrak{h}\circ c)'(t))^2\left(1 - \frac{1}{2}\frac{\partial^2[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right) +$$
$$\left. + 2(\mathfrak{h}\circ c)'(t)\frac{\partial[\dot{c}(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))\right] =$$
$$= -\frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}\left[((\mathfrak{h}\circ c)'(t))^2\left((\dot{x}(t))^2 - 1 + (x(t))^2 + \frac{3(y(t))^2}{(\mathfrak{h}(c(t)))^4}\right) + \right.$$
$$\left. +4(\mathfrak{h}\circ c)'(t)\left(\mathfrak{h}(c(t))\dot{x}(t)x(t) - \frac{\dot{y}(t)y(t)}{(\mathfrak{h}(c(t)))^3}\right) + \frac{(\dot{y}(t))^2}{((\mathfrak{h}\circ c)'(t))^2}\right].$$

For concreteness let

$$c(t) = (x(t), y(t)) = (t\cos\alpha, \sqrt{2} + t\sin\alpha), \text{ and } t_0 = 0.$$

Then we have that $(\mathfrak{h}(c(t_0)))^2 = 2$ because in the formula

$$\frac{1 \pm \sqrt{1 + 4(1 - x(t)^2)y(t)^2}}{2(1 - x(t)^2)}$$

we have to calculate with positive sign. Since

$$(\mathfrak{h}\circ c)'(t_0) = \sqrt{2}\frac{\sqrt{2}\sin\alpha}{4+2} = \frac{1}{3}\sin\alpha,$$

we get that

$$\mathrm{I} = \frac{1}{9}\sin^2\alpha(\cos^2\alpha - 1) + \frac{\sin^2\alpha}{\frac{1}{9}\sin^2\alpha} = 9 - \frac{1}{9}\sin^4\alpha > 0.$$

Similarly the second fundamental form is

$$\mathrm{II} = -\frac{2}{3}\left(\frac{1}{9}\sin^2\alpha\left(\cos^2\alpha - 1 + \frac{3}{2}\right) + 9 + \frac{2\sqrt{2}}{3}\sin^2\alpha\right) =$$
$$= -\frac{2}{3}\left(\left(\frac{1}{6} + \frac{2\sqrt{2}}{3}\right)\sin^2\alpha - \frac{1}{9}\sin^4\alpha + 9\right) =$$
$$= -\frac{1 + 4\sqrt{2}}{9}\sin^2\alpha + \frac{2}{27}\sin^4\alpha - 6.$$

The extremal values of the non-positive function

$$\frac{\mathrm{II}}{\mathrm{I}} = \frac{\frac{2}{27}\sin^4\alpha - \frac{1+4\sqrt{2}}{9}\sin^2\alpha - 6}{9 - \frac{1}{9}\sin^4\alpha}$$

attained at the directions α for which either $\cos\alpha = 0$ or $\sin\alpha = 0$ with the respective negative values $-\frac{157+12\sqrt{2}}{240}$ and $-\frac{2}{3}$. Since the normal vector at this point is

$$n\circ c = \frac{2\mathfrak{h}(c(t))}{2\mathfrak{h}(c(t)) - \frac{\partial[c(t),c(t)]^\tau}{\partial\tau}(\mathfrak{h}(c(t)))}c(t) + (\mathfrak{h}\circ c)(t)e_3 = \frac{2}{3}(0,\sqrt{2})^T + \sqrt{2}e_3 =$$
$$= \sqrt{2}\left(\left(0, \frac{2}{3}\right)^T + e_3\right),$$

we have that the norm-square of it is $2\left(\frac{2}{9} - 1\right) = -\frac{14}{9} < 0$ and hence the Gaussian curvature is negative at this point.

(2) For a further example choose an ellipse G_α as in the previous example with a fixed parameter α, where $1 \leq \alpha \leq 2$. Let $K(\tau)$ be the rotated copy of this ellipse about the time axis with the angle τ. Then

$$[s,s]^\tau = [xe_1+ye_2, xe_1+ye_2]^\tau = \alpha^2(\cos\tau x + \sin\tau y)^2 + \frac{(-\sin\tau x + \cos\tau y)^2}{\alpha^2} =$$
$$= \left(\alpha^2x^2+\frac{y^2}{\alpha^2}\right)\cos^2\tau + \left(\alpha^2y^2+\frac{x^2}{\alpha^2}\right)\sin^2\tau + 2\cos\tau\sin\tau\left(\alpha^2-\frac{1}{\alpha^2}\right) =$$
$$= \left(\alpha^2x^2+\frac{y^2}{\alpha^2}\right) + \left(\alpha^2-\frac{1}{\alpha^2}\right)\left(y^2-x^2\right)\sin^2\tau + 2\cos\tau\sin\tau\left(\alpha^2-\frac{1}{\alpha^2}\right) =$$
$$= \left(\alpha^2x^2+\frac{y^2}{\alpha^2}\right) + \left(\alpha^2-\frac{1}{\alpha^2}\right)\left(y^2-x^2\right)\frac{1}{2} -$$
$$-\frac{1}{2}\left(\alpha^2-\frac{1}{\alpha^2}\right)\left(y^2-x^2\right)\cos 2\tau + \sin 2\tau\left(\alpha^2-\frac{1}{\alpha^2}\right) =$$
$$= \frac{1}{2}(\alpha^2+\frac{1}{\alpha^2})(x^2+y^2) + \left(\alpha^2-\frac{1}{\alpha^2}\right)\left(\sin 2\tau - \frac{1}{2}\left(y^2-x^2\right)\cos 2\tau\right)$$

The implicite equation of the imaginary unit sphere is

$$\tau = \sqrt{1+\frac{\alpha^4+1}{2\alpha^2}(x^2+y^2) + \frac{\alpha^4-1}{\alpha^2}\left(\sin 2\tau - \frac{1}{2}\left(y^2-x^2\right)\cos 2\tau\right)}.$$

Here there is no explicit form for τ however in a concrete point the fundamental forms and curvatures can be determined. We remark that the Hausdorff distances of the unit ball $K(\tau)$ from B_E is less or equal to 1, thus the domain is the whole plane. Since the norm induced by an inner product in every moments, the corresponding time-space is a semi-Riemann manifold.
(3) We can get premanifolds if the square of the examined norms can not be represented as the scalar square of an inner product. A three-dimensional example can be get from the function $K(\tau)$ which sends τ for $\tau > 1$ to the unit ball of the l_τ space with Euclidean area π. In this case

$$[s,s]^\tau = \frac{v(l_\tau)}{\pi}\sqrt[\tau]{|x|^\tau+|y|^\tau}, \text{ where } v(l_\tau) = \frac{\Gamma\left(1+\frac{1}{\tau}\right)^2}{\Gamma\left(1+\frac{2}{\tau}\right)}4$$

is the volume of the unit ball of the standard l_τ norm of the plane. Here for τ we have the implicite equality

$$\tau = \sqrt{1+\frac{v(l_\tau)}{\pi}\sqrt[\tau]{|x|^\tau+|y|^\tau}}.$$

As in the previous example the domain is also the plane S. □

3.4.1.2. *The de Sitter sphere in time-space.* The points of the de Sitter sphere $G^{+,T}$ can be defined by the union $\cup\left\{\left\{s+\tau e_n \text{ where } \sqrt{[s,s]^\tau-1}=\tau\right\}, [s,s]^\tau \geq 1\right\}$. G^+ is not a hypersurface. It can be handled by the implicit function $\tau = \sqrt{-1+[s,s]^\tau}$ for $[s,s]^\tau > 1$, using the assumption $\frac{\partial\mathfrak{G}}{\partial\tau}(v_0) = \frac{\frac{\partial([s,s]^\tau)}{\partial\tau}}{2\sqrt{[s,s]^\tau-1}}(v_0) - 1 \neq 0$, or equivalently $\frac{\partial([s_0,s_0]^\tau)}{\partial\tau}(\tau_0) \neq 2\sqrt{[s_0,s_0]^{\tau_0}-1}$. Using the equality $\mathfrak{h}^2(s)+\mathfrak{g}^2(s) = [s,s]^{\mathfrak{h}(s)} + [s,s]^{\mathfrak{g}(s)}$, the derivative of $\mathfrak{g}$ in the direction of the unit vector $e \in S$ can be calculated from the equality

$$2\mathfrak{h}(s)\mathfrak{h}'_e(s) + 2\mathfrak{g}(s)\mathfrak{g}'_e(s) = \left([s,s]^{\mathfrak{h}(s)} + [s,s]^{\mathfrak{g}(s)}\right)' =$$
$$= \left(2[e,s]^{\mathfrak{h}(s)} + \frac{\partial[s,s]^{\mathfrak{h}(s)}}{\partial\tau}(\tau)\cdot\mathfrak{h}'_e(s)\right) + \left(2[e,s]^{\mathfrak{g}(s)} + \frac{\partial[s,s]^{\mathfrak{g}(s)}}{\partial\tau}(\tau)\cdot\mathfrak{g}'_e(s)\right).$$

Thus

$$\mathfrak{g}'_e(s) = \frac{2[e,s]^{\mathfrak{g}(s)}}{2\mathfrak{g}(s) - \frac{\partial [s,s]^{\mathfrak{g}(s)}}{\partial \tau}(\mathfrak{g}(s))}.$$

The first and second fundamental forms have analogous forms as in the case of the imaginary unit sphere $H^{+,T}$.

3.4.1.3. *The shape function.* To use our new model in relativity theory we can clarify the following question: *How we define the so-called "inertial frame" in our model?* If we insist on "a Descartes-system of the space which moving with a constant velocity" then we have to interpret two things; the concepts of Descartes system and the concept of velocity, respectively. In a deterministic time-space we have a function $K(\tau)$, and we have more possibilities to define orthogonality in a concrete moment τ. We shall fixe a concept of orthogonality and we will consider it in every normed space. The most natural choice is the concept of Birkhoff orthogonality. Using it, in every normed space we can consider an Auerbach basis (see Theorem 3.1.2) which can play the role of a basic coordinate frame. We can determine the coordinates of the point with respect to this basis. We say that a frame is *at rest with respect to the absolute time* if its origin (as a particle) is at rest with respect to the absolute time τ and the unit vectors of its axes are at rest with respect to a fixed Euclidean orthogonal basis of S. In this case the world line of the origin in the model is a vertical line (parallel to T); it is the collection of those points of the model which absolute space-coordinates do not changes by the change of the absolute time. Unfortunately, practically we do not know an absolute coordinate system, and we can not check the immobility of the axes of such a frame. This motivates our definition on inertial frame and inertial frame "at rest", respectively. We denote by $(S, \|\cdot\|^\tau)$ the normed space with unit ball $K(\tau)$. In S we fix an Euclidean orthonormal basis and give the coordinates of a point (vector) of S with respect to this basis. We get curves in S parameterized by the time τ. In our concept the particle is a random function $x : I_x \to S$ holding two conditions:

- the set $I_x \subset T^+$ is an interval
- $[x(\tau), x(\tau)]^\tau < 0$ if $\tau \in I_x$.

The particle lives on the interval I_x, is born at the moment $\inf I_x$ and dies at the moment $\sup I_x$. Since all time-sections of a time-space model is a normed space of dimension n the Borel sets of the time-sections are independent from the time. This means that we can consider the physical specifies of a particle as a trajectory of a stochastic process. A particle "realistic" if it holds the "known laws of physic" and "idealistic" otherwise. This is only a terminology for own use, the mathematical contain of the expression "known laws of physics" is indeterminable. Since the norm (and thus the metric) in a time-space model changes by the time, the formulas of the density function of a fixed distribution also changes by the time. For example, if we say that both of the functions $f(x(\tau_1))$ and $f(x(\tau_2))$ have normal distribution on its domain $\tau_1 K(\tau_1)$ and $\tau_2 K(\tau_2)$ we have to use distinct formulas on their density functions, respectively. The uniform distribution is the only distribution which density function is independent from the time. First we introduce an inner metric $\delta_{K(\tau)}$ on the space at the moment τ.[3] These thread motivates the following definition:

DEFINITION 3.4.3 ([11]). *Let $X(\tau) : T \to \tau K(\tau)$ be a continuously differentiable (by the time) trajectory of the random function $(x(\tau)\ ,\ \tau \in I_x)$. We say that the particle $x(\tau)$ is* realistic in its position *if for every $\tau \in I_x$ the random variable $\delta_{K(\tau)}(X(\tau), x(\tau))$ has normal distribution*

[3]We have two possibilities, either we can consider this space with its original metric $\delta_{K(\tau)}(u,v) := \|u-v\|^\tau$, (arise from the norm) – at this time the space bounded and all distances are less then 2τ – or as another possibility we can define a distance which derives from the ball $\tau K(\tau)$ indirectly. For example let $u, v \in \tau K(\tau)$ be two points and denote by $(uv)_\infty$ and $(uv)_{-\infty}$ the intersection points of the line (uv) and the boundary of the ball $\tau K(\tau)$, respectively. (Here the point v separates the points u and $(uv)_\infty$.) Let $(u, v, (uv)_\infty, (uv)_{-\infty})$ denote the cross ratio of the four points and let $\delta_{K(\tau)}(u,v) := \ln(u, v, (uv)_\infty, (uv)_{-\infty})$ be the inner metric of the space $\tau K(\tau)$. We note that if the norm is Euclidean it is the usual distance of a modeled hyperbolic space (which is unbounded with respect to this metric).

on $\tau K(\tau)$. In other words the stochastic process $\left(\delta_{K(\tau)}\left(X(\tau), x(\tau)\right)\ ,\ \tau \in I_x\right)$ has stationary Gaussian process with respect to a given continuously differentiable function $X(\tau)$. We call the function $X(\tau)$ the world-line *of the particle $x(\tau)$.*

We note that the two metrics defined in footnote 2 are essentially agree for small distances, thus the concept of "realistic in its position" independent from the choice of $\delta_{K(\tau)}$. As a refinement of this concept we define another one, which can be considered as a generalization of the principle on the maximality of the speed of the light.

DEFINITION 3.4.4 ([11]). *We say that a particle* realistic in its speed *if it is realistic in its position and the derivatives of its world-line $X(\tau)$ are time-like vectors.*

Since the shape of the sets of the time-like points in a time-space is not a cone, it is possible that u is a time-like vector but αu is not with certain α. On the other hand in a random time-space model the speed of those particles which realistic in its speed with a great probability are less than to the speed of the light. Note that our theory does not exclude the possibility of the existence of a particle with speed is greater to the speed of the light at a moment neither in the case of generalized space-time model or in the case of a particle which is realistic in its speed. For such two particles x', x'' which are realistic in their position we can define a instantaneously distance by the equality:

$$\delta(x'(\tau), x''(\tau)) = \|X'(\tau) - X''(\tau)\|^{\tau} = \sqrt{[X'(\tau) - X''(\tau), X'(\tau) - X''(\tau)]^{+,T}}.$$

We can say that two particles x' and x'' are agree if the expected value of their distances is equal to zero. Let $I = I_{x'} \cap I_{x''}$ be the common part of their domains. The required equality is:

$$E(\delta_{K(\tau)}(x'(\tau), x''(\tau))) = \int\limits_I \delta_{K(\tau)}(x'(\tau), x''(\tau))\mathrm{d}\tau = \int\limits_I \|X'(\tau) - X''(\tau)\|^{\tau}\mathrm{d}\tau = 0.$$

We also define the concept of a frame as follows:

DEFINITION 3.4.5 ([11]). *The system $\{f_1(\tau), f_2(\tau), f_3(\tau), o(\tau)\} \subset (S, \|\cdot\|^{+\tau}) \times \tau K(\tau)$ is a* frame, *if the following assumptions hold:*

- *$o(\tau)$ is a particle realistic in its speed, with such a world-line $O(\tau) : T \to \tau K(\tau)$ which does not intersect the absolute time axis T,*
- *the functions $f_i(\tau) : T \to \cup\{(S, \|\cdot\|^{\tau})\ ,\ \tau \in T\}$ are continuously differentiable, for all fixed τ,*
- *the system $\{f_1(\tau), f_2(\tau), f_3(\tau)\}$ is an Auerbach basis with origin $O(\tau)$ in the normed space $(S, \|\cdot\|^{\tau})$.*

We remark that a condition stating that the frame building up from elements of an Auerbach basis is very strong. In the most cases the Auerbach basis is unique. In an inner product space a set of pairwise conjugate diameters of element n of the unit ellipsoid gives an Auerbach basis. It is easy to see that every two Auerbach basis are isometric to each other, there is a linear isometry of the space sending the first into the second. Thus the set of the Auerbach bases can be get using the elements of the symmetry group of the space from a fixed one. The following lemma is obviously and we leave its proof to the reader.

LEMMA 3.4.2 ([11]). *For every $\varepsilon > 0$ and a pair $\{K', \mathcal{A}'\}$ where $K' \in \mathcal{K}_0$ is a unit ball of C^2-class and $\mathcal{A}'$ is an Auerbach basis of the normed space $(S, \|\cdot\|_{K'})$ there is a $\delta > 0$ such that if for K'' holds $\delta_H(K', K'') < \delta$ then it can be found an Auerbach basis $\mathcal{A}'' \in (S, \|\cdot\|_{K''})$ for which $\delta_H(\mathcal{A}', \mathcal{A}'') < \varepsilon$ holds.*

Note, that in a good model we have to guarantee that Einstein's convention on the equivalence of the inertial frames can be remained for us. However at this time we have no possibility to give the concepts of "frame at rest" and the concept of "frame which moves constant velocity with respect to another one". The reason is that when we changed the norm of the space by the function $K(\tau)$ we concentrated only the change of the shape of the unit ball and did not use any

correspondence between the points of the two unit balls. Obviously, in a concrete computation we should proceed vice versa, first we should give a correspondence between the points of the old unit ball and the new one and this implies the change of the norm. To this purpose we may define a homotopic mapping $\mathbf{K}$ which describes the deformation of the norm. From Lemma 3.4.2 above it follows that we can define a shape function as follows

DEFINITION 3.4.6. [11] *The homotopic mapping* $\mathbf{K}(x,\tau) : (S, \|\cdot\|_E) \times T \to (S, \|\cdot\|_E)$ *is called by the* shape function of the time-space *if it holds the following assumptions:*

- $\mathbf{K}(x,\tau)$ *is homogeneous in its first variable and continuously differentiable in its second one,*
- $\mathbf{K}(\{e_1,e_2,e_3\},\tau)$ *is an Auerbach basis of* $(S, \|\cdot\|^\tau)$ *for every* τ,
- $\mathbf{K}(B_E,\tau) = K(\tau)$.

The shape function determines the changes at all levels in a time space, for example a frame is "at rest" if its change arises only from this globally determined change, and "moves with constant velocity" if its origin has this property and the directions of its axes are "at rest". Precisely, we say, that

DEFINITION 3.4.7 ([11]). *The frame* $\{f_1(\tau), f_2(\tau), f_3(\tau), o(\tau)\}$ moves with constant velocity with respect to the time-space *if for every pairs* τ, τ' *in* T^+ *we have*

$$f_i(\tau) = \mathbf{K}(f_i(\tau'),\tau) \quad \text{for all } i \text{ with } 1 \le i \le 3$$

and there are two vectors $O = o_1e_1 + o_2e_2 + o_3e_3 \in S$ *and* $v = v_1e_1 + v_2e_2 + v_3e_3 \in S$ *that for all values of* τ *we have* $O(\tau) = \mathbf{K}(O,\tau) + \tau\mathbf{K}(v,\tau)$. *A frame is* at rest with respect to the time-space *if the vector* v *is the zero vector of* S.

Consider the derivative of the above equality by τ. We get that $\dot{O}(\tau) = \frac{\partial \mathbf{K}(O,\tau)}{\partial\tau} + \mathbf{K}(v,\tau) + \tau\frac{\partial \mathbf{K}(v,\tau)}{\partial\tau}$, showing that for such a homotopic mapping, which is constant in the time, the orbit of $O(\tau)$ is a line with direction vector v through the origin of the time space. Similarly in the case when v is the zero vector it is a vertical (parallel to T) line-segment through O.

EXAMPLE 3.4.2. [11] Consider the second example of Example 3.4.1. The shape function can be get as follows: $\mathbf{K}\left((x,y)^T,\tau\right) = \left(\alpha x\cos\tau - \frac{1}{\alpha}y\sin\tau, \alpha x\sin\tau + \frac{1}{\alpha}y\cos\tau\right)^T$. Then we have $\mathbf{K}(B_E,\tau) = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} G_\alpha$ furthermore we get also that $\mathbf{K}(e_1,\tau) = (\alpha\cos\tau, \alpha\sin\tau)^T$, $\mathbf{K}(e_2,\tau) = \left(-\frac{1}{\alpha}\sin\tau, \frac{1}{\alpha}\cos\tau\right)^T$ gives an Auerbach basis for the corresponding norm. The unit vectors of a frame at rest can be get if we use the affinity $\begin{pmatrix} \alpha\cos\tau & \frac{1}{\alpha}\sin\tau \\ -\alpha\sin\tau & \frac{1}{\alpha}\cos\tau \end{pmatrix}$ for the vectors $(\cos\beta,\sin\beta)^T$, $(-\sin\beta,\cos\beta)^T$, respectively. (Here β is a given parameter.) With respect to the absolute coordinate-system the world-line of the origin is a helical

$$\tau \mapsto \left(\alpha o_1\cos\tau + \frac{1}{\alpha}o_2\sin\tau, -\alpha o_1\sin\tau + \frac{1}{\alpha}o_2\cos\tau\right)^T$$

through a given point $O = (o_1,o_2)^T$ of the plane S. □

The concept of shape function gives a chance to define the so-called time-axes.

DEFINITION 3.4.8 ([11]). *A* time-axis *of the time-space model is a world-line* $O(\tau)$ *of such a particle which moves with constant velocity with respect to the time-space and starts from the origin. More precisely, for the world-line* $(O(\tau),\tau)$ *we have* $\mathbf{K}(O,\tau) = 0$ *and hence with a given vector* $v \in S$, $O(\tau) = \tau\mathbf{K}(v,\tau)$.

EXAMPLE 3.4.3. Let the function $\mathbf{K}$ is defined (as in the previous example) with the equality:

$$\mathbf{K}\left((x,y)^T,\tau\right) = \left(\alpha x\cos\tau - \frac{1}{\alpha}y\sin\tau, \alpha x\sin\tau + \frac{1}{\alpha}y\cos\tau\right)^T,$$

then the time-axis defined by the vector $v = (v_1, v_2)^T$ is the curve

$$\left(\tau\left(\alpha v_1 \cos\tau - \frac{1}{\alpha}v_2 \sin\tau\right), \tau\left(\alpha v_1 \sin\tau + \frac{1}{\alpha}v_2 \cos\tau\right), \tau\right)^T. \quad \square$$

In a space-time model a time-axis is a line through the origin. Moreover the time-axes intersect the imaginary unit sphere orthogonally. In time-space this is not true generally.

PROPOSITION 3.4.1. [11] *A time-axis intersects orthogonally of the imaginary sphere of parameter c at the point $(s, \tau^\star)$ if and only if for all directions e of S with the function $c(t+\lambda) : \lambda \to s + \lambda e$ holds the equality:*

$$\left[\dot{c}(t), c(t) + \frac{1}{c^2 - \|v\|_E^2}\frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}\left(\frac{1}{\sqrt{c^2 - \|v\|_E^2}}\right)\right]^{\frac{1}{\sqrt{c^2-\|v\|_E^2}}} =$$

$$= \left[\dot{c}(t), \frac{\frac{2c^2}{\sqrt{c^2-\|v\|_E^2}}}{\frac{2c^2}{\sqrt{c^2-\|v\|_E^2}} - \frac{\partial [c(t),c(t)]^\tau}{\partial \tau}\left(\frac{1}{\sqrt{c^2-\|v\|_E^2}}\right)}c(t)\right]^{\frac{1}{\sqrt{c^2-\|v\|_E^2}}}.$$

Before the proof we observe that if the shape function does not depend on the time that the required equality holds.

PROOF. The time-axis and the imaginary sphere of parameter c intersect in the point at the parameter value $\tau^\star$. Thus we have $(\tau^\star)^2\left([\mathbf{K}(v,\tau^\star), \mathbf{K}(v,\tau^\star)]^{\tau^\star} - c^2\right) = -1$ or reordering it the other one: $[\mathbf{K}(v,\tau^\star), \mathbf{K}(v,\tau^\star)]^{\tau^\star} = \left(c^2 - \frac{1}{(\tau^\star)^2}\right)$. We note that for an arbitrary pair v and τ we have the equality $[\mathbf{K}(v,\tau), \mathbf{K}(v,\tau)]^{\tau} = \|v\|_E^2 [\mathbf{K}(v^0,\tau), \mathbf{K}(v^0,\tau)]^{\tau} = \|v\|_E^2$, where v^0 is the unit vector in the direction of v. Hence $\|v\|_E^2 = \left(c^2 - \frac{1}{(\tau^\star)^2}\right)$ or equivalently $(\tau^\star)^2 = \frac{1}{c^2-\|v\|_E^2}$. Now we determine the angle of the imaginary unit sphere and the time-axis defined above. The velocity vector of the time-axis at the examined point is

$$\tau^\star \mathbf{K}(v,\tau^\star) + (\tau^\star)^2 \frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}(\tau^\star) + \tau^\star e_4.$$

If we recalculate the tangent vector of the unit sphere of parameter c at its point $s + \tau e_4$ using also the opportunity $c(t+\lambda) = s + \lambda e$, we get that it is

$$\dot{c}(t) + \frac{2\left[\dot{c}(t), c(t)\right]^\tau}{2c^2\tau - \frac{\partial [c(t),c(t)]^\tau}{\partial \tau}(\tau)}e_4$$

The product of the two vectors is

$$\left[\dot{c}(t), \tau^\star \mathbf{K}(v,\tau^\star) + (\tau^\star)^2 \frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}(\tau^\star)\right]^{\tau^\star} - c^2 \frac{2\tau^\star \left[\dot{c}(t), c(t)\right]^{\tau^\star}}{2\tau^\star c^2 - \frac{\partial [c(t),c(t)]^\tau}{\partial \tau}(\tau^\star)} =$$

$$= \left[\dot{c}(t), \tau^\star \mathbf{K}(v,\tau^\star) + (\tau^\star)^2 \frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}(\tau^\star)\right]^{\tau^\star} - \left[\dot{c}(t), \frac{2c^2\tau^\star c(t)}{2\tau^\star c^2 - \frac{\partial [c(t),c(t)]^\tau}{\partial \tau}(\tau^\star)}\right]^{\tau^\star}.$$

Since we have $\tau^\star \mathbf{K}(v,\tau^\star) = s = c(t)$ this formula can be simplified into the form

$$\left[\dot{c}(t), c(t) + (\tau^\star)^2 \frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}(\tau^\star)\right]^{\tau^\star} - \left[\dot{c}(t), \frac{2c^2\tau^\star}{2\tau^\star c^2 - \frac{\partial [c(t),c(t)]^\tau}{\partial \tau}(\tau^\star)}c(t)\right]^{\tau^\star}.$$

$\square$

A non-trivial situation in which the above orthogonality holds if for the unknown function $\alpha(\tau^\star)$ the following equation system can be solved:

$$\begin{aligned}\frac{\partial \mathbf{K}(v,\tau)}{\partial \tau}(\tau^\star) &= \alpha(\tau^\star)c(t)\\ 1+(\tau^\star)^2\alpha(\tau^\star) &= \frac{2c^2\tau^\star}{2\tau^\star c^2 - \frac{\partial [c(t),c(t)]^\tau}{\partial\tau}(\tau^\star)}.\end{aligned}$$

In fact, if we eliminate $\alpha(\tau^\star)$ we get the following equation:

$$(\tau^\star)^2\left(2\tau^\star c^2 - \frac{\partial\left[c(t),c(t)\right]^\tau}{\partial\tau}(\tau^\star)\right)\frac{\partial \mathbf{K}(v,\tau)}{\partial\tau}(\tau^\star) = \frac{\partial\left[c(t),c(t)\right]^\tau}{\partial\tau}(\tau^\star)c(t).$$

Example 3.4.4. In this example we show that there is non-trivial shape function for which the above equality on orthogonality holds. Let define the shape function by the non-zero scalar valued function $\mathbf{K}(v,\tau) = \alpha(v,\tau)v$. Then we get that $\frac{\partial \mathbf{K}(v,\tau)}{\partial\tau} = \frac{\partial\alpha(v,\tau)}{\partial\tau}v$ and $\mathbf{K}(c(t),\tau) = \alpha(c(t),\tau)c(t)$, implying the equality $\alpha^2(c(t),\tau)\left[c(t),c(t)\right]^\tau = \|c(t)\|_E^2$. Since $\alpha(v,\tau)\neq 0$, from $[c(t),c(t)]^\tau = \frac{\|c(t)\|_E^2}{\alpha^2(c(t),\tau)}$ we get that

$$\frac{\partial\left[c(t),c(t)\right]^\tau}{\partial\tau} = -\frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\frac{\partial\alpha(c(t),\tau)}{\partial\tau}.$$

The orthogonality condition for a general τ means the equality

$$\tau^2\left(2\tau c^2 + \frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\frac{\partial\alpha(c(t),\tau)}{\partial\tau}\right)\frac{\partial\alpha(c(t),\tau)}{\partial\tau}v = -\frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\frac{\partial\alpha(c(t),\tau)}{\partial\tau}c(t)$$

and again if the function $\alpha(v,\tau)$ is a constant we have a solution. In the other case, we can simplify it with its derivative and get that

$$(\tau)^2\left(2\tau c^2 + \frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\right)\frac{\partial\alpha(c(t),\tau)}{\partial\tau}v = -\frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}c(t).$$

We also know the connection between $c(t)$ and v, because at the point $\tau^\star$ we have $c(t) = \tau^\star\mathbf{K}(v,\tau^\star) = \tau^\star\alpha(v,\tau^\star)v$. This simplifies the above equality to equality among scalar functions:

$$(\tau)^2\left(2\tau c^2 + \frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\right)\frac{\partial\alpha(c(t),\tau)}{\partial\tau} = -\frac{2\|c(t)\|_E^2}{\alpha^3(c(t),\tau)}\tau^\star\alpha(c(t),\tau^\star),$$

which can be written in the form

$$-\frac{\tau^3c^2}{\tau^2+\tau^\star\alpha(c(t),\tau^\star)} = \frac{\frac{\partial\alpha(c(t),\tau)}{\partial\tau}}{\alpha^3(c(t),\tau)}.$$

Solving this separable differential equation, we get the following solution

$$\alpha^2(c(t),\tau) = \frac{(\tau^\star)^2\alpha^2(c(t),\tau^\star)\|v\|_E^2}{c^2\left(\tau^2-\tau^\star\alpha(c(t),\tau^\star)\ln(\tau^2+\tau^\star\alpha(c(t),\tau^\star))\right) + (\tau^\star)^2\alpha^2(c(t),\tau^\star)\|v\|_E^2C}.$$

To get the identity at the point $\tau^\star$ we substitute it and we can determine the constant C.

$$C = \frac{(\tau^\star)^2\left(\|v\|_E^2-c^2\right) + c^2\tau^\star\alpha(c(t),\tau^\star)\ln((\tau^\star)^2+\tau^\star\alpha(c(t),\tau^\star))}{(\tau^\star)^2\alpha^2(c(t),\tau^\star)\|v\|_E^2}.$$

With this constant the required equality on $\alpha(c(t),\tau)$ is

$$\alpha^2(c(t),\tau) = \frac{(\tau^\star)^2\alpha^2(c(t),\tau^\star)\|v\|_E^2}{c^2\tau^2 - (\tau^\star)^2\left(c^2-\|v\|_E^2\right) - c^2\tau^\star\alpha(c(t),\tau^\star)\ln\left(\frac{\tau^2+\tau^\star\alpha(c(t),\tau^\star)}{(\tau^\star)^2+\tau^\star\alpha(c(t),\tau^\star)}\right)}.$$

The function $\alpha(c(t),\tau)$ is well-defined real valued function if the right hand side is greater or equal to zero. From this assumption we get the inequality

$$\tau^2 - \tau^\star\alpha(c(t),\tau^\star)\ln\left(\tau^2+\tau^\star\alpha(c(t),\tau^\star)\right) \geq$$

$$\geq \left(1 - \frac{\|v\|_E^2}{c^2}\right)(\tau^\star)^2 - \tau^\star \alpha(c(t),\tau^\star)\ln\left((\tau^\star)^2 + \tau^\star\alpha(c(t),\tau^\star)\right).$$

Since the left hand side is a monotone increasing function of its variable $\tau \geq 0$, we have to pick up a value in which the equality holds to determine a range interval where this equality also holds. It is easy to calculate that at the value

$$\tau = \sqrt{\left(1 - \frac{\|v\|_E^2}{c^2}\right)}\tau^\star$$

the equality holds thus $\alpha^2(c(t),\tau)$ can be defined well if $\tau \geq \sqrt{\left(1 - \frac{\|v\|_E^2}{c^2}\right)}\tau^\star$.

Using the assumption that the point $c(t)$ is on the imaginary sphere of parameter c we get that

$$\alpha(c(t),\tau^\star)^2 = c^2\tau^{\star 2} - 1,$$

and thus

$$\alpha^2(c(t),\tau) = \frac{(\tau^\star)^2(c^2\tau^{\star 2}-1)\|v\|_E^2}{c^2\tau^2 - \tau^{\star 2}\left(c^2 - \|v\|_E^2\right) - \tau^\star\sqrt{c^2(\tau^\star)^2-1}\ln\left(\frac{\tau^2+\tau^\star\sqrt{c^2(\tau^\star)^2-1}}{(\tau^\star)^2+\tau^\star\sqrt{c^2(\tau^\star)^2-1}}\right)}.$$

□

3.4.2. Random time-space model. Of course, we can choose the function $K(\tau)$ "randomly". To this purpose we use Kolmogorov's extension theorem (or theorem on consistency, see in [99]). This says that a suitably "consistent" collection of finite-dimensional distributions will define a probability measure on the product space. The sample space here is $\mathcal{K}_0$ with the Hausdorff distance. It is a locally compact, separable (second-countable) metric space. By Blaschke's selection theorem (see in [78]) $\mathcal{K}$ is a boundedly compact space so it is also complete. It is easy to check that $\mathcal{K}_0$ is also a complete metric space if we assume that the non-proper bodies (centrally symmetric convex compact sets with empty interior) also belong to it. Let P be such a probability measure which defined in Subsection 3.3.2. In every moment we consider the same probability space $(\mathcal{K}_0, P)$ and also consider in each of the finite collections of moments the corresponding product spaces $((\mathcal{K}_0)^r, P^r)$. The consistency assumption of Kolmogorov's theorem now automatically holds. By the extension theorem we have a probability measure $\hat{P}$ on the measure space of the functions on T to $\mathcal{K}_0$ with the σ-algebra generated by the cylinder sets of the space. The distribution of the projection of $\hat{P}$ to the probability space of a fix moment is the distribution of P.

DEFINITION 3.4.9 ([11]). *Let $(K_\tau\ ,\ \tau \geq 0)$ be a random function defined as an element of the Kolmogorov's extension $\left(\Pi\mathcal{K}_0, \hat{P}\right)$ of the probability space $(\mathcal{K}_0, P)$. We say that the generalized space-time model with the random function*

$$\hat{K}_\tau := \sqrt[n]{\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(K_\tau)}}K_\tau$$

is a random time-space model. *Here $\alpha_0(K_\tau)$ is a random variable with truncated normal distribution and thus $(\alpha_0(K_\tau)\ ,\ \tau \geq 0)$ is a stationary Gaussian process. We call it the* shape process *of the random time-space model.*

It is clear that a deterministic time-space model is a special trajectory of the random time-space model. The following theorem is essential.

THEOREM 3.4.2 ([11]). *For a trajectory $L(\tau)$ of the random time-space model, for a finite set $0 \leq \tau_1 \leq \cdots \leq \tau_s$ of moments and for a $\varepsilon > 0$ there is a deterministic time-space model defined by the function $K(\tau)$ for which*

$$\sup_i\{\rho_H\left(L(\tau_i), K(\tau_i)\right)\} \leq \varepsilon.$$

PROOF. Since the set of centrally symmetric convex bodies with C^∞-boundary is dense in the set of centrally symmetric convex bodies (see [135]), we can choose, for every τ_i, a body $K(\tau_i) \in \mathcal{K}_0$ with C^2 boundary with the required volume for which $\rho_H(L(\tau_i), K(\tau_i)) \leq \varepsilon$ holds. We prove that these bodies can be connected with such a trajectory of the random time-space model for which the function K holds the properties of the defining function of a deterministic time-space model. The impact of the K function on a fixed vector $s \in S$ can be checked on the vary of its norm. Using the Minkowski functional, we can get the norm of a vector s as the length of a fixed segment relative to the length of the diameter of the unit ball intersected by the half-line containing the segment $[O, P]$. This means that we can determine the change of the length of a diameter of a fixed direction if we change the shape of the body by the time. Consider a representation of the body by polar coordinates with respect to its center O. Since the boundary of the body is of class C^2, all of their coordinate functions have the analogous property. This function depends also on the time τ, the change of the unit ball implies the change of its coordinate functions. We say that the trajectory $K(\tau)$ is a continuously differentiable function if for a fixed coordinate representation its coordinate functions are continuously differentiable functions of the time. This is equivalent to the property that the support function $h_{(K(\tau))}$ is continuously differentiable as the function of the time τ. The differentiability property of the trajectory implies the analogous differentiability property of the change of the norm of a fix vector since the points of the boundary of the unit ball has an equation of the form $r^\tau = (r(\varphi_1, \cdots, \varphi_{n-1}))^\tau$. We can conclude that if the trajectory $K(\tau)$ is a continuously differentiable function, this holds also for the function $\tau \to \sqrt{[s,s]^\tau}$. In a space S with an inner product the polarity equation implies the required assumption. If S is (only) a smooth normed space with a semi inner product, we need further comments. Since for a differentiable norm function McShane's equality holds, we have

$$[x,y]^\tau = \|y\|^\tau((\|\cdot\|^\tau)'_x(y)) = \|y\|^\tau(\|\cdot\|'_x(y))^\tau.$$

On the other hand, the function $(\|\cdot\|'_x(y))^\tau$ is also continuously differentiable function of y, thus the thread using on the norm function above is applicable for it, too. This means that the differentiability property of the trajectory implies the analogous differentiability property of the function $\tau \to (\|\cdot\|'_x(y))^\tau$. Using the rule of the product function we also have that $\tau \to [x,y]^\tau$ is continuously differentiable if the trajectory $\tau \to K(\tau)$ holds this property.
We now define a differentiable trajectory through the points $(\tau_i, K(\tau_i))$. If $\tau, \tau'_i \in [\tau_i, \tau_{i+1}]$ denote by $K_{Bezier}(\tau)$ the formal Bezier spline of second order through the points $(\tau_i, K(\tau_i))$ and $(\tau_{i+1}, K(\tau_{i+1}))$ with "tangents" through the point $(\tau'_i, L(\tau'_i))$. Thus we have by definition

$$K_{Bezier}(\tau) := \left(1 - \frac{\tau - \tau_i}{\tau_{i+1} - \tau_i}\right)^2 K(\tau_i) + 2\left(1 - \frac{\tau - \tau_i}{\tau_{i+1} - \tau_i}\right)\frac{\tau - \tau_i}{\tau_{i+1} - \tau_i} L(\tau_i) + \left(\frac{\tau - \tau_i}{\tau_{i+1} - \tau_i}\right)^2 K(\tau_{i+1}),$$

where the addition is the Minkowski addition and the product is the respective homothetic mapping. If we assume that for all values of i $(1 < i < s)$ the body $K(\tau_i)$ is a Minkowski convex combination of the bodies $L(\tau'_i)$ and $L(\tau'_{i+1})$ the function $K_{Bezier}(\tau)$ is valid on the whole interval $[\tau_1, \tau_s]$. Since for positive constants α, β we have

$$h_{\alpha K' + \beta K''}(x) = \alpha h_{K'}(x) + \beta h_{K''}(x),$$

we also get that $K_{Bezier}(\tau)$ is a continuously differentiable trajectory in its whole domain. We have to prove yet that for a fixed τ, the set $K_{Bezier}(\tau)$ is a centrally symmetric convex compact body with C^2-class boundary but these statements follow immediately from the concept of Minkowski linear combination.
Finally we normalize this trajectory under the volume function and extract it to the whole T. The function $K(\tau)$ determines a required deterministic time-space model if we define it as

follows:

$$K(\tau) = \begin{cases} \sqrt[n]{\frac{\text{vol}(B_E)}{\text{vol}(K_{Bezier}(\tau_s))}} K_{Bezier}(\tau_s) & \text{if} \quad \tau_s < \tau \\ \sqrt[n]{\frac{\text{vol}(B_E)}{\text{vol}(K_{Bezier}(\tau))}} K_{Bezier}(\tau) & \text{if} \quad \tau_1 \leq \tau \leq \tau_s \\ \sqrt[n]{\frac{\text{vol}(B_E)}{\text{vol}(K_{Bezier}(\tau_1))}} K_{Bezier}(\tau_1) & \text{if} \quad \tau < \tau_1 \ . \end{cases}$$

□

An important consequence of this theorem that without loss of generality we can assume, that the time-space model is deterministic.

APPENDIX A

Relativity theory in time-space

Our model – described in the previous section – can be considered also as a model of the universe[1]. The deterministic variant obviously contains as a special case the model of Minkowski space-time. On the other hand it can be extended to a generalization of the Robertson-Walker space-time, too. The advantage of our model that S can be considered also as a general normed space (without inner product).
The time-space can be defined in a more convenient way, using a shape function. It regulates the methods of calculations in time-space and gives the possibility to rewrite the equality of special and global relativity.

A.1. On the formulas of special relativity theory

Consider the upper part of the imaginary sphere of parameter c in a four-dimensional deterministic time-space model. Without the imaginary unit sphere we consider the imaginary unit sphere H_c of parameter c with the corresponding product $[x', x'']^{+,T} := [s', s'']^{\tau''} + c^2 [\tau', \tau'']$. Practically the constant c can be considered as the speed of the light in vacuum. Assume that the shape-function is a two-times continuously differentiable function. We need two axioms to interpret in time-space of the usual axioms of special relativity theory. First we assume that:

AXIOM A.1.1. *The laws of physics are invariant under transformations between frames. The laws of physics will be the same whether you are testing them in frame "at rest", or a frame moving with a constant velocity relative to the "rest" frame.*

AXIOM A.1.2. *The speed of light in a vacuum is measured to be the same by all observers in frames.*

These two axioms can be transformed into the language of the time-space by the method of Minkowski [**123**]. To this we use H_c introduced and the group G_c as the set of those isometries of the space which leave invariant H_c. Such an isometry can be interpreted as a coordinate transformation of the time-space which sends the axis of the absolute time into another time-axis t', and also maps the intersection point of the absolute time-axis with the imaginary sphere H_c into the intersection point of the new time-axis and H_c. An isometry of the time-space is also a homeomorphism thus it maps the subspace S into a topological hyperplane S' of the embedding normed space. S' is orthogonal to the new time-axis in the sense that its tangent hyperplane at the origin is orthogonal to t' with respect to the product of the space. Of course the new space-axes are continuously differentiable curves in S' which tangents at the origin are orthogonal to each other. Since the absolute time-axis is orthogonal to the imaginary sphere H_c the new time-axis t' must holds this property, too. Thus the investigations in the previous section are essential from this point of view. Assuming that the definition of the time-space implies this property we can get some formulas similar to of special relativity. We note that the function $\mathbf{K}(v, \tau)$ holds the orthogonality property of vectors of S and by the equality $[\mathbf{K}(v, \tau), \mathbf{K}(v, \tau)]^{\tau} = \|v\|_E^2$ we can see also that the formulas on time-dilatation and length-contraction are valid, too. This implies that using the well-known notations $\beta = \frac{\|v\|_E}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ we get that the connection between the time τ_0 and τ of an event measuring by two observers one of at rest and the other moves with an constant velocity $\|v\|_E$ with respect to the time-space is $\tau = \gamma\tau_0$. Consider now a moving rod which points move constant velocity with respect to the time-space such that it is always parallel to the velocity vector $\mathbf{K}(v, \tau)$. Then we have $\|v\|_E = \frac{L_0}{T}$ where T is the time calculated from the length L_0 and the velocity vector v by such an observer which moves with the rod. Another observer can calculate the length L from the measured time T_0 and the velocity v by the formula $\|v\|_E = \frac{L}{T_0}$. Using the above formula of dilatation we get the known Fitzgerald contraction of the rod: $L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma}$.
Lorentz transformation in time space also based on the usual experiment in which we send a ray of light to a mirror in direction of the unit vector e with distance d from me.
If we at rest we can determine in time space the points A, C and B of departure, turn and arrival of the ray of light, respectively. A and B are on the absolute time-axis at heights τ_A, and τ_B, respectively. The position of C is

$$(\tau_C - \tau_A)\mathbf{K}(ce, \tau_C - \tau_A) + \tau_C e_4 = \frac{\tau_B - \tau_A}{2}\mathbf{K}\left(ce, \frac{\tau_B - \tau_A}{2}\right) + \frac{\tau_B + \tau_A}{2}e_4,$$

[1] In this appendix we check the usability our concept in practice. Despite the content of this appendix belongs to the area of theoretical physics it is strongly connected to the useless of my mathematical investigations.

since we know that the light take the road back and forth over the same time. We observe that the norm of the space-like component s_C is $\|s_C\|^{\tau_C} = c\frac{\tau_B-\tau_A}{2}$ as in the usual case of space-time.
The moving observer synchronized its clock with the observer at rest in the origin, and moves in the direction v with velocity $\|v\|_E$. We assume that the moving observer also sees the experiment thus its time-axis corresponding to the vector v meats the world-line of the light in two points A' and B' positioning on the respective curves AC and CB. This implies that the respective space-like components of the world-line of the light and the world-line of the axis are parallels to each other in every minutes. By formula we have: $\|v\|_E\mathbf{K}(e,\tau) = \mathbf{K}(v,\tau)$. From this we get the equality $\tau_{A'}\mathbf{K}(v,\tau_{A'}) + \tau_{A'}e_4 = (\tau_{A'}-\tau_A)\mathbf{K}(ce,\tau_{A'}-\tau_A) + \tau_{A'}e_4$. This implies that ${\tau_{A'}}^2\|v\|_E{}^2 - c^2{\tau_{A'}}^2 = (\tau_{A'}-\tau_A)^2c^2 - c^2{\tau_{A'}}^2$ and thus $\tau_{A'} = \frac{c}{c-\|v\|_E}\tau_A$. The proper time $(\tau_{A'})_0$ is $(\tau_{A'})_0 = \sqrt{1-\beta^2}\frac{c}{c-\|v\|_E}\tau_A = \tau_A\sqrt{\frac{1+\beta}{1-\beta}}$. Similarly we also get that $(\tau_{B'})_0 = \tau_B\sqrt{\frac{1-\beta}{1+\beta}}$, and we determine the new time coordinate of the point C with respect to the new coordinate system:

$$(\tau_C)_0 = \frac{(\tau_{A'})_0 + (\tau_{B'})_0}{2} = \frac{1}{2}\left(\tau_A\sqrt{\frac{1+\beta}{1-\beta}} + \tau_B\sqrt{\frac{1-\beta}{1+\beta}}\right).$$

Since we have that the norm of the space-like component is $\|s_C\|_E = c\frac{\tau_B-\tau_A}{2}$, we get that $\tau_A = \tau_C - \frac{\|s_C\|_E}{c}$ and $\tau_B = \tau_C + \frac{\|s_C\|_E}{c}$ and thus

$$(\tau_C)_0 = \frac{1}{2}\left(\left(\tau_C - \frac{\|s_C\|_E}{c}\right)\sqrt{\frac{1+\beta}{1-\beta}} + \left(\tau_C + \frac{\|s_C\|_E}{c}\right)\sqrt{\frac{1-\beta}{1+\beta}}\right) = \frac{\tau_C - \frac{\beta\|s_C\|_E}{c}}{\sqrt{1-\beta^2}} =$$
$$= \frac{\tau_C - \frac{\|v\|_E\|s_C\|_E}{c^2}}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}} = \frac{\tau_C - \frac{[\mathbf{K}(s_C,\tau_C),\mathbf{K}(v,\tau_C)]^{\tau_C}}{c^2}}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}.$$

On the other hand we also have that the space-like component $((s_C)_0)_S$ of the transformed space-like vector $(s_C)_0$ arise also from a vector parallel to e thus it is of the form $\mathbf{K}(((s_C)_0)_S,\tau) = \|((s_C)_0)_S\|_E\mathbf{K}(e,\tau)$. For the norm of $(s_C)_0$ we know that $\|(s_C)_0\|^{+,T} = c\frac{(\tau_{B'})_0-(\tau_{A'})_0}{2}$, hence $\|(s_C)_0\|^{+,T} = \frac{\|s_C\|_E-\|v\|_E\tau_C}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}$. If we consider the vector $\widehat{(s_C)_0} = \gamma\left(\mathbf{K}(s_C,\tau_C) - \mathbf{K}(v,\tau_C)\tau_C\right) \in S$, we get a norm-preserving, bijective mapping $\widehat{L}$ from the world-line of the light into S with the definition

$$\widehat{L} : \mathbf{K}((s_C)_0,(\tau_C)_0) \mapsto \gamma\left(\mathbf{K}(s_C,\tau_C) - \mathbf{K}(v,\tau_C)\tau_C\right).$$

The connection between the space-like coordinates of the point with respect to the two frames now has a more familiar form. Henceforth the Lorentz transformation means for us the correspondence:

$$\begin{aligned} s &\mapsto \widehat{\mathbf{K}(s',\tau')} = \gamma\left(\mathbf{K}(s,\tau) - \mathbf{K}(v,\tau)\tau\right) \\ \tau &\mapsto \tau' = \gamma\left(\tau - \frac{[\mathbf{K}(s,\tau),\mathbf{K}(v,\tau)]^{\tau}}{c^2}\right), \end{aligned}$$

and the inverse Lorentz transformation the another one

$$\begin{aligned} \widehat{\mathbf{K}(s',\tau')} &\mapsto \mathbf{K}(s,\tau) = \gamma\left(\mathbf{K}(s',\tau') + \mathbf{K}(v,\tau')\tau'\right) \\ \tau' &\mapsto \tau = \gamma\left(\tau' + \frac{[\mathbf{K}(s',\tau'),\mathbf{K}(v,\tau')]^{\tau'}}{c^2}\right). \end{aligned}$$

First note that we can determine the components of $(s_C)_0$ with respect to the absolute coordinate system, too. Since $(s_C)_0$ and $\tau\mathbf{K}(v,\tau) + \tau e_4$ are orthogonal to each other we get that

$$[\mathbf{K}(((s_C)_0)_S,\tau_C),\mathbf{K}(v,\tau_C)]^{\tau_C} = c^2((s_C)_0)_T,$$

implying that $((s_C)_0)_T = \frac{\|((s_C)_0)_S\|_E\|v\|_E}{c^2}$. Thus we get the equality

$$\|((s_C)_0)_S\|_E^2\left(1 - c^2\left(\frac{\|v\|_E}{c^2}\right)^2\right) = \left(\frac{\|s_C\|_E - \|v\|_E\tau_C}{\sqrt{1-\frac{\|v\|_E^2}{c^2}}}\right)^2,$$

implying that

$$\|((s_C)_0)_S\|_E = \frac{\|s_C\|_E - \|v\|_E\tau_C}{\left(1-\frac{\|v\|_E^2}{c^2}\right)} = \gamma^2\left(\|s_C\|_E - \|v\|_E\tau_C\right)$$

and

$$((s_C)_0)_T = \frac{\|((s_C)_0)_S\|_E\|v\|_E}{c^2} = \frac{\|v\|_E\|s_C\|_E - \|v\|_E^2\tau_C}{c^2 - \|v\|_E^2}.$$

We get that

$$(s_C)_0 = \gamma^2 \left(\|s_C\|_E - \|v\|_E \tau_C\right) \left(\mathbf{K}(e, \tau_C) + \frac{\|v\|_E}{c^2} e_4\right) =$$

$$= \gamma^2 \left(\mathbf{K}(s_C, \tau_C) - \mathbf{K}(v, \tau_C)\tau_C\right) + \left(\frac{\gamma}{1-\gamma}\right)^2 \left(\|s_C\|_E - \|v\|_E \tau_C\right) e_4.$$

We can determine also the length of this vector in the new coordinate system, too. Since

$$[(s_C)_0, (s_C)_0]^{+,T} = \left(\|(s_C)_0\|^{+,T}\right)^2 = \frac{(\|s_C\|^{\tau_C} - \|v\|_E \tau_C)^2}{1 - \frac{\|v\|_E^2}{c^2}} = \frac{[s_C, s_C]^{\tau_C} - 2\|s_C\|^{\tau_C}\|v\|_E \tau_C + (\|v\|_E \tau_C)^2}{1 - \frac{\|v\|_E^2}{c^2}}$$

and

$$((\tau_C)_0)^2 = \frac{(\tau_C)^2 - 2\tau_C \frac{\|v\|_E \|s_C\|^{\tau_C}}{c^2} + \frac{(\|v\|_E \|s_C\|^{\tau_C})^2}{c^4}}{1 - \frac{\|v\|_E^2}{c^2}},$$

hence the equality $[(s_C)_0, (s_C)_0]^{+,T} - c^2 ((\tau_C)_0)^2 = [s_C, s_C]^{\tau_C} - c^2 (\tau_C)^2$ shows that under the action of the Lorentz transformation the "norm-squares" of the vectors of the time-space are invariant as in the case of the usual space-time.

Finally we determine those points of the space which new time-coordinates are zero and thus we get a mapping from the subspace S into the time-space. Let $s \in S$ arbitrary and consider the corresponding point $\mathbf{K}(s, \tau) + \tau e_4$ and assume that $0 = \tau_0 = \gamma\tau - \gamma \frac{\|v\|_E}{c^2} \|\mathbf{K}(s, \tau)\|^\tau$, hence $\tau = \frac{\|v\|_E \|s\|_E}{c^2}$. Then we get the mapping of the coordinate subspace S under the action of the isometry corresponding to that Lorentz transformation which sends the absolute time-axis into the time-axis $\tau\mathbf{K}(v, \tau) + \tau e_4$ in question. This is the set

$$S_0 = \left\{ \mathbf{K}\left(s, \frac{\|v\|_E \|s\|_E}{c^2}\right) + \frac{\|v\|_E \|s\|_E}{c^2} e_4 \quad | \quad s \in S \right\}.$$

For a boost in an arbitrary direction with velocity v, it is convenient to decompose the spatial vector s into components perpendicular and parallel to v: $s = s_1 + s_2$ so that $[\mathbf{K}(s, \tau), \mathbf{K}(v, \tau)]^\tau = [\mathbf{K}(s_1, \tau), \mathbf{K}(v, \tau)]^\tau + [\mathbf{K}(s_2, \tau), \mathbf{K}(v, \tau)]^\tau = [\mathbf{K}(s_2, \tau), \mathbf{K}(v, \tau)]^\tau$. Then, only time and the component $\mathbf{K}(s_2, \tau)$ in the direction of $\mathbf{K}(v, \tau)$;

$$\begin{aligned} \tau' &= \gamma \left(\tau - \frac{[\mathbf{K}(s, \tau), \mathbf{K}(v, \tau)]^\tau}{c^2}\right) \\ \widehat{\mathbf{K}(s', \tau')} &= \mathbf{K}(s_1, \tau) + \gamma(\mathbf{K}(s_2, \tau) - \mathbf{K}(v, \tau)\tau) \end{aligned}$$

are "distorted" by the Lorentz factor γ. The second equality can be written also in the form:

$$\widehat{s'} = \mathbf{K}(s, \tau) + \left(\frac{\gamma - 1}{\|v\|_E^2} [\mathbf{K}(s, \tau), \mathbf{K}(v, \tau)]^\tau - \gamma\tau\right) \mathbf{K}(v, \tau).$$

REMARK A.1.1. *If we have two time-axes $\tau\mathbf{K}(v', \tau) + \tau e_4$ and $\tau\mathbf{K}(v'', \tau) + \tau e_4$ then there are two subgroups of the corresponding Lorentz transformations mapping the absolute time-axis onto another time-axes, respectively. These two subgroups are also subgroups of G_c. Their elements can be paired on the base of their action on S. The pairs of these isometries define a new isometry of the space (and its inverse) on a natural way, with the composition one of them and the inverse of the other. Omitting the absolute time-axis from the space (as we suggest earlier) the invariance of the product on the remaining space and also the physical axioms of special relativity can remain in effect.*

If $\mathbf{K}(u, \tau)$ and $\mathbf{K}(v, \tau')$ are two velocity vectors then using the formula for inverse Lorentz transformation of the corresponding differentials we get that $\mathrm{d}\tau = \gamma \left(\mathrm{d}\tau' + \frac{[\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'}}{c^2}\right)$ and $\mathbf{K}(\mathrm{d}s, \mathrm{d}\tau) = \mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau') + \left(\frac{1-\gamma}{\|v\|_E^2} [\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'} + \gamma \mathrm{d}\tau'\right) \mathbf{K}(v, \tau')$. Thus

$$\mathbf{K}(u, \tau) = \frac{\mathbf{K}(\mathrm{d}s, \mathrm{d}\tau)}{\mathrm{d}\tau} = \frac{\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau') + \left(\frac{1-\gamma}{\|v\|_E^2} [\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'} + \gamma \mathrm{d}\tau'\right) \mathbf{K}(v, \tau')}{\gamma \left(\mathrm{d}\tau' + \frac{[\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'}}{c^2}\right)} =$$

$$= \frac{\left(\mathbf{K}(v, \tau') + \frac{1}{\gamma} \frac{\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau')}{\mathrm{d}\tau'} + \frac{1+\gamma}{\gamma c^2} \left[\frac{\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau')}{\mathrm{d}\tau'}, \mathbf{K}(v, \tau')\right]^{\tau'} \mathbf{K}(v, \tau')\right)}{1 + \frac{\left[\frac{\mathbf{K}(\mathrm{d}\widehat{s'}, \mathrm{d}\tau')}{\mathrm{d}\tau'}, \mathbf{K}(v, \tau')\right]^{\tau'}}{c^2}}$$

$$= \frac{\left(\mathbf{K}(v, \tau') + \frac{1}{\gamma} \mathbf{K}(u', \mathrm{d}\tau') + \frac{1+\gamma}{\gamma c^2} [\mathbf{K}(u', \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'} \mathbf{K}(v, \tau')\right)}{1 + \frac{[\mathbf{K}(u', \mathrm{d}\tau'), \mathbf{K}(v, \tau')]^{\tau'}}{c^2}}.$$

Our following starting point is *the velocity vector (or four-velocity)*. The absolute time coordinate is τ, this defines a world line of form $S(\tau)=\mathbf{K}(s(\tau),\tau)+\tau e_4$. Its proper time is $\tau_0=\frac{\tau}{\gamma}=\tau\sqrt{1-\frac{\|v\|_E^2}{c^2}}$, where v is the velocity vector of the moving frame. By definition

$$V(\tau):=\frac{\mathrm{d}S(\tau)}{\mathrm{d}\tau_0}=\gamma\left(\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+e_4\right).$$

If the shape-function is a linear mapping then $\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}=\mathbf{K}(\dot{s}(\tau),1):=\mathbf{K}(v(\tau),1)$ and we also have $[V(\tau),V(\tau)]^{+,T}=\gamma^2\left([\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1-c^2\right)=-c^2$. The *acceleration* is defined as the change in four-velocity over the particle's proper time. Hence now the velocity of the particle is also a function of τ as without γ we have the function $\gamma(\tau)$. The definition is:

$$A(\tau):=\frac{\mathrm{d}V}{\mathrm{d}\tau_0}=\gamma(\tau)\frac{\mathrm{d}V}{\mathrm{d}\tau}=\gamma^2(\tau)\frac{\mathrm{d}^2\mathbf{K}(s(\tau),\tau)}{\mathrm{d}\tau^2}+\gamma(\tau)\gamma'(\tau)\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+\gamma(\tau)\gamma'(\tau)e_4,$$

where with notation $a(\tau)=v'(\tau)=s''(\tau)$,

$$\gamma'(\tau)=\left(\frac{1}{\sqrt{1-\frac{\|v(\tau)\|_E^2}{c^2}}}\right)'=\left(\frac{1}{\sqrt{1-\frac{[\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1}{c^2}}}\right)'=\frac{\left[\frac{\mathrm{d}(\mathbf{K}(v(\tau),1)}{\mathrm{d}\tau},\mathbf{K}(v(\tau),1)\right]^1}{c^2\left(1-\frac{[\mathbf{K}(v(\tau),1),\mathbf{K}(v(\tau),1)]^1}{c^2}\right)^{\frac{3}{2}}}=$$

$$=\frac{\left[\frac{\mathrm{d}(\mathbf{K}(v(\tau),1)}{\mathrm{d}\tau},\mathbf{K}(v(\tau),1)\right]^1}{c^2}\gamma^3(\tau),$$

In the case of linear shape-function it has the form $A(\tau)=\gamma^2(\tau)\mathbf{K}(a(\tau),0)+\gamma(\tau)\gamma'(\tau)\mathbf{K}(v(\tau),1))+\gamma(\tau)\gamma'(\tau)e_4$. Since in this case $[V(\tau),V(\tau)]^{+,T}=-c^2$, we have

$$[A(\tau),V(\tau)]^{T,+}=\gamma^3(\tau)\left([\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1+\right.$$

$$\left.+\gamma^2(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1}{c^2}\|v(\tau)\|_E^2-\gamma^2(\tau)\left[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)\right]^1\right)=$$

$$=\gamma^3(\tau)\left([\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^1-\frac{c^2-\|v(\tau)\|_E^2}{c^2-\|v(\tau)\|_E^2}\left[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)\right]^1\right)=0.$$

By Theorem 2 on the derivative of the product (corresponding to smooth and strictly convex norms) we also get this result, in fact we have

$$0=\frac{\mathrm{d}[V(\tau),V(\tau)]^{+,T}}{\mathrm{d}\tau}=2\left[\frac{\mathrm{d}V}{\mathrm{d}\tau},V\right]^{+,T}+\frac{\partial[V(\tau),V(\tau)]^{\tau}}{\partial\tau}(1)\cdot 0=\frac{2}{\gamma}[A(\tau),V(\tau)]^{+,T}.$$

Also in the case of linear shape-function the *momentum* is $P=m_0V=\gamma m_0\left(\mathbf{K}(v(\tau),\tau)+e_4\right)$ where m_0 is the invariant mass. We also have that $[P,P]^{+,T}=\gamma^2m_0^2(\|v\|_E^2-c^2)=(m_0c)^2$. Similarly the *force* is $F=\frac{\mathrm{d}P}{\mathrm{d}\tau}=m_0\gamma^2(\tau)\mathbf{K}(a(\tau),\tau)+\gamma(\tau)\gamma'(\tau)\mathbf{K}(v(\tau),\tau))+\gamma(\tau)\gamma'(\tau)e_4$, and thus holds $[F,V]^{+,T}=0$.

A.2. General relativity theory

In time-space there is a way to describe and visualize certain spaces which are solutions of Einstein's equation. The first method is when we embed into an at least four-dimensional time-space as an four-dimensional manifold which inner metric is a solution of the Einstein equation. Our basic references here are the books [50] and [70].

A.2.1. Metrics embedded into a time-space.

A.2.1.1. *The Minkowski-Lorentz metric.* The simplest example of a Lorentz manifold is the *flat-space metric* which can be given as $\mathbb{R}^4$ with coordinates (t,x,y,z) and the metric function: $\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$. In the above coordinates, the matrix representation is $\eta=\begin{pmatrix}-c&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$. In spherical coordinates (t,r,θ,ϕ), the flat space metric takes the form $\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}r^2+r^2\mathrm{d}\Omega^2$.

It can be considered also in a 5-dimensional time-space with shape-function $\mathbf{K}(v,\tau)=v$ as the metric of a 4-dimensional subspace through the absolute time-axis. By the equivalence of time axes in a usually space-time it also can be considered as arbitrary subspace distinct to the 4-dimensional subspace of space-like vectors, too.

A.2.1.2. *The de Sitter and the anti-de Sitter metrics.* The *de Sitter space* is the space defined on the de Sitter sphere of a Minkowski space of one higher dimension. Usually the metric can be considered as the restriction of the Minkowski metric $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ to the sphere $-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = \alpha^2 = \frac{3}{\Lambda}$, where Λ is the cosmological constant (see e.g. in [70]). Using also our constant c this latter equation can be rewrite as $-ct^2 + (x_1')^2 + (x_2')^2 + (x_3')^2 + (x_4')^2 = 1$ where $x_0 = t$, $\frac{1}{\alpha} = c$ and $x_i' = \frac{1}{\alpha}x_i$. This shows that in the 5-dimensional time space with shape-function $\mathbf{K}(v,\tau) = v$ it is the hyperboloid with one sheet with circular symmetry about the absolute time-axis.
The *anti-de Sitter space* is the hyperbolic analogue of the elliptic de Sitter space. The Minkowski space of one higher dimension can be restricted to the so called *anti-de Sitter sphere* (also called by in our terminology as imaginary sphere) defined by the equality $-x_0^2+x_1^2+x_2^2+x_3^2 = -\alpha^2$. The shape function again is $\mathbf{K}(v,\tau) = v$ and the corresponding 4-submanifold is the hyperboloid of two sheets with hyperplane symmetry as the 4-subspace S of space-time vectors.

A.2.1.3. *The Friedmann-Lemaître-Robertson-Walker metrics.* A standard metric forms of the Friedmann-Lemaître-Robertson-Walker metrics (F-L-R-W) family of space-times can be obtained by using suitable coordinate parameterizations of the 3-spaces of constant curvature. One of its forms is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R^2(t)}{1+\frac{1}{4}k(x^2+y^2+z^2)}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

where $k \in \{-1,0,1\}$ is fixed. By the parametrization $\tau = t$ this metric is the metric of a time-space with shape-function $\mathbf{K}(v,\tau)$. Observe that $\|v\|_E^2 = [\mathbf{K}(v,\tau),\mathbf{K}(v,\tau)]^\tau = \frac{R^2(\tau)}{1+\frac{1}{4}k\|v\|_E^2}\|\mathbf{K}(v,\tau)\|_E^2$. Note that we can choose the constant k also as a function of the absolute time τ giving a deterministic time-space with more generality. Hence the shape-function is $\mathbf{K}(v,\tau) = \frac{\sqrt{1+\frac{1}{4}k(\tau)\|v\|_E^2}}{R(\tau)}v$.

A.2.2. Three-dimensional visualization of a metric in a four-time-space. The second method is when we consider a four-dimensional time-space and a three-dimensional sub-manifold in it with the property that the metric of the time-space at the points of the sub-manifold can be corresponded to the given one. This method gives a good visualization of the solution in a case when the examined metric has some speciality e.g. there is no dependence on time or (and) the metric has a spherical symmetry. The examples of this section are also semi-Riemannian manifolds. We consider now such solutions which have the form:

$$\mathrm{d}s^2 = -(1-f(r))c^2\mathrm{d}t^2 + \frac{1}{1-f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$$

where $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$ is the standard metric on the 2-sphere. Thus we have to search a shape function $\mathbf{K}(v,\tau)$ of the embedding space and a sub-manifold of it on which the Minkowski-metric gives the required one. If the metric isotropic we have a chance to give it by isotropic coordinates. To this we substitute the parameter r by the function $r = g(r^\star)$, and solve the differential equation:

$$f(g(r^\star)) = 1 - \left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$$

for the unknown function $g(r^\star)$. Then we get the metric in the isotropic form

$$\mathrm{d}s^2 = -\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2 c^2\mathrm{d}t^2 + \frac{g^2(r^\star)}{r^{\star 2}}\left(\mathrm{d}r^{\star 2} + r^{\star 2}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\right).$$

For isotropic rectangular coordinates $x = r^\star\sin\theta\cos\phi$, $y = r^\star\sin\theta\sin\phi$ and $z = r^\star\cos\theta$ the metric becomes

$$\mathrm{d}s^2 = -\left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2 c^2\mathrm{d}t^2 + \frac{g^2(r^\star)}{r^{\star 2}}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

where $r^\star = \sqrt{x^2+y^2+z^2}$. From this substituting $ds^2 = 0$ and rearranging the equality, we get that the velocity of the light is

$$\sqrt{\frac{\mathrm{d}x^2}{\mathrm{d}t^2}+\frac{\mathrm{d}y^2}{\mathrm{d}t^2}+\frac{\mathrm{d}z^2}{\mathrm{d}t^2}} = \frac{r^{\star 2}g'(r^\star)}{g^2(r^\star)}c,$$

independent from its direction and varies with only the radial distance $r^\star$ (from the point mass at the origin of the coordinates). In the points of the hypersurface $t = r^\star = \sqrt{x^2+y^2+z^2}$ the metric can be parameterized by the time:

$$\mathrm{d}s^2 = -\left(\frac{tg'(t)}{g(t)}\right)^2 c^2\mathrm{d}t^2 + \frac{g^2(t)}{t^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right),$$

and from the equation

$$\frac{tg'(t)}{g(t)}\mathrm{d}t = \mathrm{d}\tau$$

we can give a re-scale of the time by the parametrization

$$\tau := \int t\frac{g'(t)}{g(t)}\mathrm{d}t = t\ln(g(t)) - \int \ln(g(t))\mathrm{d}t.$$

From this equation we determine the inverse function $\hat{g}$ for which $t = \hat{g}(\tau)$. Since $\hat{g}(\tau) = t = r^\star = \sqrt{x^2+y^2+z^2}$ we also have that the examined set of points of the space-time is a hypersurface defined by the equality:

$$\tau = \left(t\ln(g(t)) - \int \ln(g(t))dt\right)\sqrt{x^2+y^2+z^2}.$$

This implies a new form of the metric at the points of this hypersurface:

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right).$$

The corresponding inner product has the matrix form: $\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} & 0 & 0 \\ 0 & 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} & 0 \\ 0 & 0 & 0 & \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2} \end{pmatrix}$ and hence the Euclidean lengthes of the vectors of the space depend only on the absolute moment τ in which we would like to measure it. Thus we can visualize the examined metric as a metric at the points of the hypersurface $\tau = \left(t\ln(g(t)) - \int \ln(g(t))\mathrm{d}t\right)\|v\|_E$ of certain time-space. We note that this is not the inner metric of the examined surface of dimension 3 which can be considered as metric of a three-dimensional space-time. To determine the shape-function observe that $\|v\|_E^2 = [\mathbf{K}(v,\tau),\mathbf{K}(v,\tau)]^\tau = \frac{g^2(\hat{g}(\tau))}{\hat{g}(\tau)^2}\|\mathbf{K}(v,\tau)\|_E^2$ from which we get that $\mathbf{K}(v,\tau) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v$. It is clear that the flat space metric can be considered in this way. Here $f(r)\equiv 0$, $g = \mathrm{id}$ and $\tau = t$ implying that $\mathbf{K}(v,\tau) = v$ and the hypersurface is the light-cone defined by $\tau = \|v\|_E$.
We now give some further examples.

A.2.2.1. *The Schwarzschild metric.* Besides the flat space metric the most important metric in general relativity is the *Schwarzschild metric* which can be given in the set of local polar-coordinates (t,r,φ,θ) by $\mathrm{d}s^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$ where, again, $\mathrm{d}\Omega^2$ is the standard metric on the 2-sphere. Here G is the *gravitation constant* and M is a constant with the dimensions of mass. The function f is $f(r) = \frac{2GM}{c^2r} := \frac{r_s}{r}$ with constant $r_s = \frac{2GM}{c^2}$. The differential equation on g is $\frac{r_s}{g(r^\star)} = 1 - \left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$ with the solution $g(r^\star) = \frac{r_s}{4}c_1r^\star\left(1+\frac{1}{c_1r^\star}\right)^2$, and if we choose $\frac{4}{r_s}$ for the parameter c_1 we get the known (see in [50]) solution $g(r^\star) = r^\star\left(1+\frac{r_s}{4r^\star}\right)^2$. For isotropic rectangular coordinates the metric becomes

$$\mathrm{d}s^2 = -\frac{(1-\frac{r_s}{4r^\star})^2}{(1+\frac{r_s}{4r^\star})^2}c^2\mathrm{d}t^2 + \left(1+\frac{r_s}{4r^\star}\right)^4(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).$$

The equation between τ and t is

$$\tau = \int\frac{(1-\frac{r_s}{4t})}{(1+\frac{r_s}{4t})}\mathrm{d}t = \int\frac{4t-r_s}{4t+r_s}\mathrm{d}t = t - 2r_s\int\frac{1}{4t+r_s}\mathrm{d}t = t - \frac{r_s}{2}\ln\left(t+\frac{r_s}{4}\right) + C.$$

Of course we can choose $C = 0$. Similarly to the known tortoise-coordinates there is no explicite inverse function of this parametrization which we denote by $\hat{g}(\tau) = t$. The shape-function of the corresponding time-space is

$$\mathbf{K}(v,\tau) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v = \left(1+\frac{r_s}{4\hat{g}(\tau)}\right)^{-2}v.$$

A.2.2.2. *The Reissner-Nordström metric.* In spherical coordinates (t,r,θ,ϕ), the line element for the Reissner-Nordström metric is $\mathrm{d}s^2 = -\left(1-\frac{r_S}{r}+\frac{r_Q^2}{r^2}\right)c^2\,\mathrm{d}t^2 + \frac{1}{1-\frac{r_S}{r}+\frac{r_Q^2}{r^2}}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$, here again t is the time coordinate (measured by a stationary clock at infinity), r is the radial coordinate, $r_S = 2GM/c^2$ is the Schwarzschild radius of the body, and r_Q is a characteristic length scale given by $r_Q^2 = \frac{Q^2G}{4\pi\varepsilon_0c^4}$. Here $1/4\pi\varepsilon_0$ is the Coulomb force constant. The function f is $f(r) = \frac{r_s}{r} - \frac{r_Q^2}{r^2}$. The differential equation on g is $\frac{r_s}{g(r^\star)} - \frac{r_Q^2}{g^2(r^\star)} = 1 - \left(\frac{r^\star g'(r^\star)}{g(r^\star)}\right)^2$ with the solution $g(r^\star) = \sqrt{\frac{r_s^2}{4}-r_Q^2}\frac{c_1}{2}r^\star\left(1+\frac{1}{c_1r^\star}\right)^2 - \sqrt{\frac{r_s^2}{4}-r_Q^2} + \frac{r_s}{2}$, if we choose $c_1 := \frac{2}{\sqrt{\frac{r_s^2}{4}-r_Q^2}}$ we get a more simple form:

$$g(r^\star) = r^\star\left(1+\frac{\sqrt{\frac{r_s^2}{4}-r_Q^2}}{2r^\star}\right)^2 - \sqrt{\frac{r_s^2}{4}-r_Q^2} + \frac{r_s}{2} = r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right) + \frac{r_s}{2}.$$

For the isotropic rectangular coordinates we have:

$$\mathrm{d}s^2 = -\left(\frac{r^\star\left(1-\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)}{r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)+\frac{r_s}{2}}\right)^2 c^2\mathrm{d}t^2 + \left(\frac{r^\star\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4r^{\star 2}}\right)+\frac{r_s}{2}}{r^\star}\right)^2 (\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2).$$

Our process now leads to the new time parameter

$$\tau = t - \left(\frac{r_s}{4}-\frac{r_Q}{2}\right)\ln\left(\left(t+\frac{r_s}{4}\right)^2-\frac{r_Q^2}{4}\right) - r_Q \ln\left(t+\frac{r_s}{4}+\frac{r_Q}{2}\right)+C,$$

which in the case of $C = r_Q = 0$ gives back the parametrization of Schwarzschild solution. The shape-function of the searched time-space can be determined by the corresponding inverse $t = \hat{g}(\tau)$, it is

$$\mathbf{K}(v,\tau) = \frac{\hat{g}(\tau)}{g(\hat{g}(\tau))}v = \frac{\hat{g}(\tau)}{\hat{g}(\tau)\left(1+\frac{\frac{r_s^2}{4}-r_Q^2}{4\hat{g}(\tau)^2}\right)+\frac{r_s}{2}}v.$$

Analogously can be computed the time-space visualization of the Schwarzschild-de Sitter solution which we now omit.

A.2.2.3. *The Bertotti-Robinson metric.* The Bertotti-Robinson space-time is the only conformally flat solution of the Einstein-Maxwell equalities for a non-null source-free electromagnetic field. The metric is: $\mathrm{d}s^2 = \frac{Q^2}{r^2}\left(-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)$, and on the light-cone $t = r$ it has the form $\mathrm{d}s^2 = -\frac{Q^2}{t^2}\mathrm{d}t^2+\frac{e^2}{t^2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)$. By the new time coordinate $\tau = Q\ln t$ or $t = e^{\frac{\tau}{Q}}$ using orthogonal space coordinates we get the form $\mathrm{d}s^2 = -\mathrm{d}\tau^2+\frac{Q^2}{e^{\frac{2\tau}{Q}}}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)$. Thus it can be visualize on the hypersurface $\tau = e\ln r$ of the time-space with shape-function: $\mathbf{K}(v,\tau) := \frac{e^{\frac{\tau}{Q}}}{Q}v$.

A.2.3. Einstein's equation. As we saw in the previous section the direct embedding of a solution of Einstein's equation into a time-space requires non-linear and very complicated shape-functions. It can be seen also that there are such solutions which there are no natural embedding into a time-space. This motivates the investigations of the present section. Our building up follows the paper of Prof. Alan Heavens [**89**].

A.2.3.1. *Homogeneous time-space-manifolds and the Equivalence Principle.* We consider now such manifolds which tangent spaces are four-dimensional time-spaces with given shape-functions. More precisely:

Definition A.2.1. *Let $\mathcal{S}$ be the set of linear mappings $\mathbf{K}(v,\tau):\mathbb{E}^3\times\mathbb{R}\longrightarrow\mathbb{E}^3$ holding the properties of a linear shape-function given in Definition 3.4.6. Giving for it the natural topology we say that $\mathcal{S}$ is* the space of shape-functions. *If we have a pair of a four-dimensional topological manifold M and a smooth (C^∞) mapping $\mathcal{K}: M\longrightarrow\mathcal{S}$ with the property that at the point $P\in M$ the tangent space is the time-space defined by $\mathcal{K}(P) = \mathbf{K}^P(s,\tau)\in\mathcal{S}$ we say that it is a* time-space-manifold. *The time-space manifold is* homogeneous *if the mapping $\mathcal{K}$ is a constant function.*

Note that a Lorentzian manifold is such a homogeneous time-space manifold which shape-function is independent from the time and it is the identity mapping on its space-like components, namely $\mathbf{K}^P(s,\tau) = s$ for all P and for all τ. Its matrix-form (using the column representation of vectors in time-space) is: $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$ Our purpose to build up the theory of global relativity in a homogeneous time-space-manifolds. We accept the so-called *Strong Equivalence Principle* of Einstein in the following form:

Axiom A.2.1. *(Equivalence Principle) At any point in a homogeneous time-space manifold it is possible to choose a* locally-inertial frame *in which the laws of physics are the same as the special relativity of the corresponding time-space.*

According to this principle, there is a coordinate-system in which a freely-moving particle moves with constant velocity with respect to the time-space $\mathcal{K}(P) = \mathbf{K}^P(s,\tau) = \mathbf{K}(s,\tau)$. It is convenient to write the world line $S(\tau) = \mathbf{K}(s(\tau),\tau)+\tau e_4$ parametrically, as a function of the proper time $\tau_0 = \frac{\tau}{\gamma(\tau)}$. In Section A.1 we determined the velocity using the time-space parameter τ:$V(\tau) = \gamma(\tau)\left(\frac{\mathrm{d}(\mathbf{K}(s(\tau),\tau))}{\mathrm{d}\tau}+e_4\right) = \gamma(\tau)\left(\mathbf{K}(v(\tau),1)+e_4\right)$. Taking into consideration again that the shape-function is linear, the acceleration is:

$$A(\tau) = \gamma^2(\tau)\mathbf{K}(a(\tau),0)+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^\tau}{c^2}\mathbf{K}(v(\tau),1)+\gamma^4(\tau)\frac{[\mathbf{K}(a(\tau),0),\mathbf{K}(v(\tau),1)]^\tau}{c^2}e_4,$$

giving the differential equation $A(\tau) = 0$ for such particle which moves linearly with respect to this frame.

A.2.3.2. *Affine connection and the metric on a homogeneous time-space manifold.* Consider any other coordinate system in which the particle coordinates are $S'(\tau_0)$. Using the chain rule, the defining equation

$$0 = A(\tau_0) = \frac{\mathrm{d}V(\tau_0)}{\mathrm{d}\tau_0} = \frac{\mathrm{d}^2S(\tau_0)}{\mathrm{d}\tau_0^2}$$

becomes

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau_0}\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\right) = \frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2} + \frac{\mathrm{d}}{\mathrm{d}\tau_0}\left(\frac{\mathrm{d}S}{\mathrm{d}S'}\right)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \frac{\mathrm{d}S}{\mathrm{d}S'}\frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2} + \frac{\mathrm{d}^2S}{\mathrm{d}S'\mathrm{d}S'}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0},$$

where $\frac{\mathrm{d}S}{\mathrm{d}S'}$ means the total derivatives of the mapping of the time-space sending the path $S'(\tau_0)$ into the specific path $S(\tau_0)$, and the trilinear function $\frac{\mathrm{d}^2S}{\mathrm{d}S'\mathrm{d}S'}$ is the second total derivatives of the same mapping. (If there is a general smooth transformation between the coordinate-frames, the corresponding derivatives are exist.) From this equality we get the tensor form of the so called *geodesic equation* of homogeneous time-space manifold, it is:

$$\frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2} + \left(\frac{\mathrm{d}S'}{\mathrm{d}S}\frac{\mathrm{d}^2S}{\mathrm{d}S'\mathrm{d}S'}\right)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2} + \Gamma(S',S)\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = 0.$$

Here we denote the inverse of the total derivatives $\frac{\mathrm{d}S}{\mathrm{d}S'}$ by $\frac{\mathrm{d}S'}{\mathrm{d}S}$. The name of $\Gamma(S',S)$ is the *affine connection.* For the uniform labelling we denote by x^4 the identity function. Since the shape function is a linear mapping we can represent it as the multiplication on left by the 3×4 matrix $K = [k_{ij}] = k^i{}_j$. In the rest of this paragraph we apply all conventions of general relativity. The Greek alphabet is used for space and time components, where indices take values 1,2,3,4 (frequently used letters are $\mu, \nu, \cdots$) and the Latin alphabet is used for spatial components only, where indices take values 1,2,3 (frequently used letters are $i, j, ...$) and according to the Einstein's convention, when an index variable appears twice in a single term it implies summation of that term over all the values of the index. The upper indices are indices of coordinates, coefficients or basis vectors.
The mapping $\mathcal{S} : S'(\tau_0) \longrightarrow S(\tau_0)$ sends $K(x'^1, x'^2, x'^3, x'^4)^T + x'^4 e_4$ into the vector $K(x^1, x^2, x^3, x^4)^T + x^4 e_4$. Denote by $\widetilde{K}$ the 4×4 matrix with coefficients: $\begin{pmatrix} k^1{}_1 & k^1{}_2 & k^1{}_3 & k^1{}_4 \\ k^2{}_1 & k^2{}_2 & k^2{}_3 & k^2{}_4 \\ k^3{}_1 & k^3{}_2 & k^3{}_3 & k^3{}_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ then we get $\mathcal{S} : \widetilde{K}(x'^1, x'^2, x'^3, x'^4)^T \mapsto \widetilde{K}(x^1, x^2, x^3, x^4)^T$. If the shape-function $\mathbf{K}$ restricted to the subspace S is a regular linear mapping than we also have $\widetilde{K}^{-1}\mathcal{S}\widetilde{K}(x'^1, x'^2, x'^3, x'^4)^T = (x^1, x^2, x^3, x^4)^T$ and we have that

$$\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right] = \frac{\mathrm{d}\widetilde{K}^{-1}\mathcal{S}\widetilde{K}}{\mathrm{d}S'} = \widetilde{K}^{-1}\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}S'}\widetilde{K} \text{ and so } \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}S'} = \widetilde{K}\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]\widetilde{K}^{-1}.$$

Hence

$$\frac{\mathrm{d}S'}{\mathrm{d}S} = \widetilde{K}\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]^{-1}\widetilde{K}^{-1} = \widetilde{K}\left[\frac{\partial x'^\mu}{\partial x^\alpha}\right]\widetilde{K}^{-1} \text{ and } \left[\frac{\mathrm{d}^2S}{\mathrm{d}S'\mathrm{d}S'}\right]^\alpha = \widetilde{K}\left[\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\right]\widetilde{K}^{-1}$$

implying that the affine connection is:

$$\Gamma(S',S)^\lambda{}_{\mu\nu} = \widetilde{K}\frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\widetilde{K}^{-1} = \widetilde{K}\Gamma^\lambda{}_{\mu\nu}\widetilde{K}^{-1} = \widetilde{K}\left\{\begin{array}{c}\lambda\\ \mu\nu\end{array}\right\}\widetilde{K}^{-1}.$$

Since $S'(\tau_0) = \widetilde{K}(x'^1, x'^2, x'^3, x'^4)^T$ thus we also get three equalities, the first one is:

$$\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \widetilde{K}\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)^T = \left(k^1{}_\alpha\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau_0}, k^2{}_\alpha\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau_0}, k^3{}_\alpha\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau_0}, k^4{}_\alpha\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau_0}\right)^T = \left[k^\lambda{}_\alpha\frac{\mathrm{d}x'^\alpha}{\mathrm{d}\tau_0}\right].$$

The second equality is:

$$\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0}\frac{\mathrm{d}S'(\tau_0)}{\mathrm{d}\tau_0} = \widetilde{K}\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)^T\left(\frac{\mathrm{d}x'^1}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^2}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^3}{\mathrm{d}\tau_0}, \frac{\mathrm{d}x'^4}{\mathrm{d}\tau_0}\right)\widetilde{K}^T = \widetilde{K}\left[\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

and the third one is:

$$\frac{\mathrm{d}^2S'(\tau_0)}{\mathrm{d}\tau_0^2} = \widetilde{K}\left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T = \left[k^\lambda{}_\alpha\frac{\mathrm{d}^2x'^\alpha}{\mathrm{d}\tau_0^2}\right].$$

The geodesic equation now:

$$0 = \widetilde{K}\left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T + \widetilde{K}\Gamma^\lambda{}_{\mu\nu}\widetilde{K}^{-1}\widetilde{K}\left[\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

or equivalently

$$0 = \left(\frac{\mathrm{d}^2x'^1}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^2}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^3}{\mathrm{d}\tau_0^2}, \frac{\mathrm{d}^2x'^4}{\mathrm{d}\tau_0^2}\right)^T + \Gamma^\lambda{}_{\mu\nu}\left[\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau_0}\frac{\mathrm{d}x'^\nu}{\mathrm{d}\tau_0}\right]\widetilde{K}^T,$$

implying that

$$0 = \frac{\mathrm{d}^2x'^\lambda}{\mathrm{d}\tau_0^2} + \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau_0}k^\nu{}_\zeta\frac{\mathrm{d}x'^\zeta}{\mathrm{d}\tau_0}.$$

Since for the proper time we have the equality

$$-c^2\mathrm{d}\tau_0^2 = \mathrm{d}S^T\begin{pmatrix}1 & 0\\ 0 & -c^2\end{pmatrix}\mathrm{d}S = \left(\frac{\mathrm{d}S}{\mathrm{d}S'}\mathrm{d}S'\right)^T\eta\frac{\mathrm{d}S}{\mathrm{d}S'}\mathrm{d}S' = \mathrm{d}S'^T g\mathrm{d}S'$$

hence

$$g(S', S) = \left(\frac{\mathrm{d}S}{\mathrm{d}S'}\right)^T\eta\frac{\mathrm{d}S}{\mathrm{d}S'}.$$

Let denote by $[{}_j{}^ik]$ the transpose of the matrix $[k^i{}_j]$ and $K^i{}_j$ the elements of the inverse of $\widetilde{K}$. Then since

$$g(S', S) = \left(\widetilde{K}^{-1}\right)^T\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]^T\widetilde{K}^T\eta\widetilde{K}\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]\widetilde{K}^{-1}$$

thus

$$g(S', S)_{\varphi\psi} = {}_\varphi{}^\mu K\frac{\partial x^\alpha}{\partial x'^\mu}{}_\alpha{}^\delta k\eta_{\delta,\varepsilon}k^\varepsilon{}_\beta\frac{\partial x^\beta}{\partial x'^\nu}K^\nu{}_\psi.$$

This matrix is the *metric tensor* of the homogeneous time-space manifold in question. If $\widetilde{K}$ is the unit matrix, then $\mu = \varphi$, $\nu = \psi$, $\alpha = \delta$ and $\beta = \varepsilon$ implying the known formula

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta}.$$

Also note that if $\widetilde{K}$ is an orthogonal transformation then we get a more simple form of the metric:

$$g(S', S) = \widetilde{K}\left[\frac{\partial x^l}{\partial x'^i}\right]^T\eta\left[\frac{\partial x^l}{\partial x'^i}\right]\widetilde{K}^T.$$

To determine the connection between the metric and the affine connection we determine the partial derivative of the metric.

$$\frac{\partial g(S', S)}{\partial x'^\lambda} = \left(\widetilde{K}^{-1}\right)^T\left[\frac{\partial^2x^\alpha}{\partial x'^\mu\partial x'^\lambda}\right]^T\widetilde{K}^T\eta\widetilde{K}\left[\frac{\partial x^\beta}{\partial x'^\nu}\right]\widetilde{K}^{-1} + \left(\widetilde{K}^{-1}\right)^T\left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]^T\widetilde{K}^T\eta\widetilde{K}\left[\frac{\partial^2x^\beta}{\partial x'^\nu\partial x'^\lambda}\right]\widetilde{K}^{-1},$$

and since

$$\frac{\partial^2x^\alpha}{\partial x'^\mu\partial x'^\lambda} = \frac{\partial x^\alpha}{\partial x'^\rho}\widetilde{K}^{-1}\Gamma(S', S)^\rho{}_{\mu\lambda}\widetilde{K}$$

we have

$$\frac{\partial g(S', S)_{\varphi\psi}}{\partial x'^\lambda} = \Gamma(S', S)^\rho{}_{\varphi\lambda}g(S', S)_{\rho\psi} + g(S', S)_{\varphi\rho}\Gamma(S', S)^\rho{}_{\lambda\psi}$$

as in the classical case. Denote by $g(S, S')^{\varphi\rho}$ the inverse of the metric tensor then we get the connection:

$$\Gamma(S', S)^\sigma{}_{\lambda\mu} = \frac{1}{2}g(S, S')^{\nu\sigma}\left\{\frac{\partial g(S', S)_{\mu,\nu}}{\partial x'^\lambda} + \frac{\partial g(S', S)_{\lambda,\nu}}{\partial x'^\mu} - \frac{\partial g(S', S)_{\mu,\lambda}}{\partial x'^\nu}\right\}.$$

Covariant derivative, parallel transport and the curvature tensor. Since we determined the affine connection we can define the *covariant derivative* of a vectors fields on the way:

$$V^\mu{}_{;\lambda} = \frac{\partial V^\mu}{\partial x'^\lambda} + \Gamma(S', S)^\mu{}_{\lambda\rho}V^\rho = \frac{\partial V^\mu}{\partial x'^\lambda} + \widetilde{K}\Gamma^\mu{}_{\lambda\delta}\widetilde{K}^{-1}V^\delta.$$

In fact, it converts vectors into tensor on the basis of the following calculation:

$$\widetilde{K}\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]\left[\frac{\partial x^\rho}{\partial x'^\lambda}\right]\widetilde{K}^{-1}V^\nu{}_{;\rho} = \widetilde{K}\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]\left[\frac{\partial x^\rho}{\partial x'^\lambda}\right]\widetilde{K}^{-1}\left(\frac{\partial V^\nu}{\partial x^\rho} + \widetilde{K}\Gamma^\nu{}_{\rho\delta}\widetilde{K}^{-1}V^\delta\right) =$$

$$= \widetilde{K}\left[\frac{\partial x'^\mu}{\partial x^\nu}\right]\left[\frac{\partial x^\rho}{\partial x'^\lambda}\right]\widetilde{K}^{-1}\left(\frac{\partial V^\nu}{\partial x^\rho} + \widetilde{K}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2x^\alpha}{\partial x'^\rho\partial x'^\delta}\widetilde{K}^{-1}V^\delta\right) =$$

$$= \frac{\partial V'^\mu}{\partial x'^\lambda} + \widetilde{K}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial^2x^\alpha}{\partial x'^\lambda\partial x'^\delta}\widetilde{K}^{-1}V'^\delta = \frac{\partial V'^\mu}{\partial x'^\lambda} + \widetilde{K}\Gamma^\mu{}_{\lambda\delta}\widetilde{K}^{-1}V'^\delta = V'^\mu{}_{;\lambda}.$$

Note that the covariant derivative of a co-vector is

$$V_{\mu;\lambda} = \frac{\partial V_\mu}{\partial x'^\lambda} - \Gamma(S', S)^\mu{}_{\lambda\rho}V^\rho,$$

and the covariant derivative of a tensor has the rule, each upper index adds a Γ term and each lower index subtracts one. For this reason the covariant derivative of the metric tensor (by our calculation above) vanishes. Again from the definition of the covariant derivative we get that the *equation of parallel transport* is now:

$$\frac{\mathrm{d}V^\mu}{\mathrm{d}\tau_0} = -\Gamma(S',S)^\mu{}_{\lambda\nu}\frac{\mathrm{d}x'^\lambda}{\mathrm{d}\tau_0}V^\nu.$$

From this it follows that the parallel-transport along a side $\delta x'^\beta$ of a small closed parallelogram is

$$\delta V^\alpha = -\Gamma^\alpha{}_{\beta\nu}(S',S)V^\nu\delta x'^\beta$$

and thus the total change around a small closed parallelogram with sides δa^μ, δb^ν is

$$\delta V^\alpha = (\Gamma^\alpha{}_{\beta\nu;\rho}(S',S)V^\nu + \Gamma^\alpha{}_{\beta\nu}(S',S)V^\nu{}_{;\rho} - \Gamma^\alpha{}_{\rho\nu;\beta}(S',S)V^\nu - \Gamma^\alpha{}_{\rho\nu}(S',S)V^\nu{}_{;\beta})\,\delta a^\beta\delta b^\rho$$

implying that $\delta V^\alpha = R(S',S)^\alpha{}_{\sigma\rho\beta}V^\sigma\delta a^\beta\delta b^\rho$. Here $R(S',S)^\alpha{}_{\sigma\rho\beta}$ is the *Riemann curvature tensor* defined by

$$R(S',S)^\alpha{}_{\sigma\rho\beta} := \Gamma(S',S)^\alpha{}_{\beta\sigma;\rho} - \Gamma(S',S)^\alpha{}_{\rho\sigma;\beta} + \Gamma(S',S)^\alpha_{\rho\nu}\Gamma(S',S)^\nu_{\sigma\beta} - \Gamma(S',S)^\alpha_{\beta\nu}\Gamma(S',S)^\nu_{\sigma\rho}.$$

The Ricci Tensor and the scalar curvature defined by

$$R(S',S)_{\sigma\beta} := R(S',S)^\alpha{}_{\sigma\alpha\beta} \text{ and } R(S',S) := R(S',S)^\sigma{}_\sigma,$$

respectively.

A.2.3.3. *Einstein's equation.* As we can saw in the previous paragraph all of the notion of global relativity can be defined in a time-space-manifold thus all of the equations between them is a well-defined equation. On the other hand Einstein's equation take into consideration the facts of physic; hence contains parameters which can not be changed. Fortunately we noted earlier that the covariant derivative of our metric tensor vanishes, too. Thus also vanishes the covariant derivative its inverse and hence we can write the Einstein's equation with *cosmological constant* Λ, too. The equation is formally the same that the original one, but contains a new (undetermined) parameter which is the matrix $\widetilde{K}$ of the shape-function. It is:

$$R(S',S)^{\mu\nu} - \frac{1}{2}g(S',S)^{\mu\nu}R(S',S) - \Lambda g(S',S)^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu},$$

where the parameter G can be adjusted so that the active and gravitational masses are equal and $T^{\mu\nu}$ is the *energy-momentum tensor*.

List of Figures

Chapter 1

Chapter 2

Chapter 3

dc_1387_17

Index

Bibliography

Papers of Á. G.Horváth:

[1] On the bisectors of a Minkowski normed space. *Acta Math. Hung.* **89(3)** (2000), 417–424
[2] Bisectors in Minkowski 3-space. *Beiträge zur Geometrie und Algebra* **45/1**, (2004) 225–238.
[3] On the shadow boundary of a centrally symmetric convex body. *Beiträge zur Geometrie und Algebra* **50/1** (2009) 219-233.
[4] Bounded representation and radial projections of bisectors in normed spaces. (common with H.Martini) *Rocky Mountain Journal of Mathematics* **43/1** (2013) 179–191.
[5] Isometries of Minkowski geometries. *Linear Algebra and Its Applications* **512**, (2017) 172-190
[6] Conics in Normed Planes. (common with H.Martini) *Extracta Math.* **26/1** (2011) 29–43.
[7] Angle measures, general rotations, and roulettes in normed planes (common with V.Balestro and H.Martini) *Analysis and Mathematical Physics* DOI: 10.1007/s13324-016-0155-3 (2016).
[8] Semi-indefinite inner product and generalized Minkowski spaces. *Journal of Geometry and Physics* **60** (2010) 1190-1208.
[9] Premanifolds. *Note di Matematica* **31/2** (2011) 17–51.
[10] Normally distributed probability measure on the metric space of norms. *Acta Mathematica Scientia* **33/5** (2013) 1231-1242.
[11] Generalized Minkowski space with changing shape. *Aequationes Mathematicae* **87/3** (2014) 337-377
[12] On the volume of the convex hull of two convex bodies (in common with Zs.Lángi) *Monatshefte für Mathematik* **174/2** (2014) 219-229.
[13] On the icosahedron inequality of László Fejes-Tóth *Journal of Mathematical Inequalities* **10/2** (2016), 521–539.
[14] Formulas on hyperbolic volume. *Aequationes Mathematicae* **83/1** (2012), 97-116.

Other papers mentioned in the Dissertation

[15] Ahn H-K., Brass P., Shin C-S., *Maximum overlap and minimum convex hull of two convex polyhedra under translations*, Comput. Geom. **40** (2008), 171-177.
[16] Alexandrov, P.S.: *Combinatorial topology.* Graylock Press Rochester, N.Y 1956.
[17] Alonso, J. and Benítez, C., *Orthogonality in normed linear spaces: a survey, II. Relations between main orthogonalities*, Extracta Math. **4** (1989), 121-131.
[18] Auerbach, H.: Sur une propiérté caractéristique de l'ellipsoide. *Studia Math.* **9** (1940) 17–22.
[19] Aurenhammer, F., Klein, R.: *Voronoi diagrams.* Handbook of Computational Geometry, Chapter 5, Eds. J.-R. Sack and J. Urrutia, North-Holland, Amsterdam, 2000, pp. 201-209.
[20] Balestro, V., G.Horváth, Á., Martini, H., Teixeira, R.: Angle in normed spaces *Aequationes Mathematicae* (2016) Doi: 10.1007/s00010-016-0445-8
[21] Bao, D., Chern S. S., Shen Z.: *An Introduction to Riemannian-Finsler Geometry.* Springer-Verlag, Berlin, 1999.
[22] Bandt, C., Baraki G.,: Metrically invariant measures on locally homogeneous spaces and hyperspaces. *Pacific. J. Math.* **121** (1986), 13–28.
[23] Bárány, I.: Affine perimeter and limit shape. *J. Reine Angew. Math.* **484** (1997), 71–84.
[24] Berkson, E.: Some types of Banach spaces, Hermitian operators and Bade functionals. *Trans. Amer. Math. Soc.* **116** (1965), 376–385.
[25] J.D. Berman and K. Hanes, *Volumes of polyhedra inscribed in the unit sphere in* E^3, Math. Ann. **188** (1970), 78–84.
[26] Birkhoff, G.: Orthogonality in linear metric spaces. *Duke Math. J.* **1** (1935), 169–172.
[27] Birkhoff G., Morris R.: Confocal conics in Space-Time. *Amer. Math. Monthly* **69(1)**, 1–4 (1962)
[28] Boltyanski, V., Martini, H., Soltan, P. S.: *Excursions into Combinatorial Geometry*, Springer, Berlin et al., 1997.
[29] Bolyai, J. *Appendix* in Tentamen written by F.Bolyai, Marosvásárhely, 1832
[30] Brass P.: Erdős distance problems in normed spaces, *Comput. Geom.* **6** (1996), 195-214

[31] P. Brass, W. Moser and J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.
[32] Brown, M.: A proof of the generalized Schoenflies theorem. *Bull.Amer. Math.Soc.* **66** (1960), 74–76.
[33] Brown, M.: Some applications of an approximation theorem on inverse limits. *Proc. Amer. Math. Soc.* **11** (1960), 478–483.
[34] Busemann, H.: Angular measure and integral curvature, *Canad. J. Math.* **1**(1949), 279-296.
[35] Busemann, H.: The geometry of Finsler spaces, *Bull. Amer. Math. Soc.* **56** (1950), 5-16.
[36] Busemann, H.: The foundations of Minkowskian geometry, *Commentarii Mathematici Helvetici* **24** (1950), 156-187.
[37] Casey, J., A treatise on spherical trigonometry, and its application to Geodesy and Astronomy, with numerous examples, Hodges, Figgis and CO., Grafton-ST. London: Longmans, Green, and CO., 1889.
[38] Clarkson, J. A.: Uniformly convex spaces. *Trans. Amer. Math. Soc.* **40** (1936), 396–414.
[39] H.T. Croft, K.J. Falconer and R.K. Guy, *Unsolved Problems in Geometry*, Vol. 2, Springer, New York, 1991.
[40] Day, M. M.: *Polygons circumscribed about closed convex curves.* Trans. Amer. Math. Soc. **62** (1947), 315–319.
[41] M.M.Day, *Normed linear spaces.* Springer-Verlag, Berlin, 1958.
[42] M.M.Day, Some characterization of inner-product spaces. *Trans. Amer. Math. Soc.* **62** (1947) 320–337.
[43] Daverman, R.J.: *Decomposition of Manifolds.* Academic Press, New York, 1986.
[44] Dekster, B. V.: An angle in Minkowski space, *J. Geom.* **80** (2004), 31-47.
[45] Dekster, B. V.: A metric space of directions in Minkowski space, *J. Geom.* **80** (2004), 48-64.
[46] Dekster, B. V.: Total angle around a point in Minkowski plane, *J. Geom.* **93** (2009), 38-45.
[47] Diminnie, C. R.: Andalafte, E.Z., Freese, R.W.: Generalized angles and a characterization of inner product spaces, *Houston J. Math.* **14** (1988), 475-480.
[48] Düvelmeyer, N.: Angle measures and bisectors in Minkowski planes, *Canad. Math. Bull.* **48/4** (2005) 523-534.
[49] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov: *Modern Geometry- Methods and Applications, Part I. The geometry of Surfaces, Transformation Groups, and Fields.* Second Edition, Springer-Verlag, 1992.
[50] Eddington, A.S. *The Mathematical Theory of Relativity* Cambridge University press, 1924.
[51] Ergin, A.A., On the one-parameter Lorentzian motion. *Comm. Fac. Sci. Univ. Ankara, Series A* **40** (1991), 59–66.
[52] Ergüt, M., Aydin, A.P., Bildik, N., The geometry of the canonical system and one-parameter motions in 2-dimensional Lorentzian space, *The Journal of Firat University* **3/1** (1988), 113–122.
[53] Fankhänel, A.: I-measures in Minkowski planes, *Beitr. Algebra Geom.* **50** (2009), 295-299.
[54] Fankhänel, A.: On angular measures in Minkowski planes, *Beitr. Algebra Geom.* **52** (2011), 335-342.
[55] Fáry, I. and Rédei, L., *Der zentralsymmetrische Kern und die zentralsymmetrische Hülle von konvexen Körpern. (German)*, Math. Ann., **122** (1950), 205-220.
[56] Federer, H.: *Geometric Measure Theory.* Springer-Verlag 1969.
[57] L. Fejes-Tóth, *Regular Figures*, The Macmillan Company, New York, 1964.
[58] Feller, W.: *An Introduction to Probability Theory and Its Applications.* 3rd ed. New York: Wiley, 1968.
[59] Finsler, P.: *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen, 1918.
[60] Finsler, P.: Über eine Verallgemeinerung des Satzes von Meusnier, *Vierteljahrsschr. Naturf. Ges. Zürich* **85** (1940), 155-164.
[61] Fleming, R. J., Jamison, J. E.: Adjoint abelian operators on L^P and $C(K)$. *Transactions of the American Math. Soc.* **217** (1976) 87–98.
[62] Garcia-Roig, J.-L.: On the group of isometries and a system of functional equations of a real normed plane, *Inner Product Spaces and Applications, Pitman Research Notes in Mathematics Series 376. Longman, Harlow* (1997), pp. 42-53.
[63] R.J. Gardner, *Geometric Tomography*, Cambridge University Press, 1995
[64] Giles, J. R.: Classes of semi inner-product spaces. *Trans. Amer. Math.Soc.* **129/3** (1967), 436–446.
[65] Glaeser, G., Stachel, H.: *Open Geometry:* OpenGL® *+ Advanced Geometry*, Springer-Verlag New York, 1999.
[66] Gohberg, I., Lancester, P., Rodman, L.: *Indefinite Linear Algebra and Applications.* Birkhauser, Basel-Boston-Berlin, 2005.
[67] Goodey P.M., Woodcock, M.M.: *Intersections of convex bodies with their translates, The Geometric Vein ed. C.Davis, B.Grünbaum and F.A.Sherk* Springer-Verlag, 1982.
[68] Goodey P.M.: Homothetic ellipsoids. *Math. Proc. Comb. Phil. Soc.* **93** (1983) 25–34.
[69] Graham R. L., Witsenhausen H. S., Zassenhaus H. J.: On tightest packings in the Minkowski plane, *Pacific Journal of Mathematics* **41/3** (1972), 699-715.
[70] Griffiths, G. P., Podolský *Exact space-times in Einstein's general relativity.* Cambridge University Press 2009.

[71] Gruber, P.M.: *Convex and Discrete Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 2007.
[72] Gruber, P.M., Wills, J.M. (Hrsg.): *Handbook of Convex Geometry* Volume A,B, North Holland, Amsterdam 1993.
[73] Gruber, P.M.: Minimal ellipsoids and their duals. *Rend. Circ. Mat. Palermo* **37/2** (1988) 35–64.
[74] Gruber P.M.: Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. I. *J. reine angew. Math.* **256** (1974) 61–83.
[75] Gruber P.M.: Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. II. *J. reine angew. Math.* **270** (1974) 123–142.
[76] Gruber P.M.: Kennzeichnende Eigenschaften von euklidischen Räumen und Ellipsoiden. III. *Monatsh. Math.* **78** (1974) 311–340.
[77] Gruber P.M., Lekkerkerker C.G.: *Geometry of numbers.* North-Holland Amsterdam-New York-Oxford-Tokyo, 1987.
[78] Gruber P.M., Wills J.M. (Hrsg.) *Handbook of Convex Geometry.* Volume A,B, North Holland, Amsterdam 1993.
[79] Gunawan, H., Lindiarni, J., Neswan, O.: P-,I-,g-, and D-angles in normed spaces, *ITB J. Sci.* **40/1** (2008), 24-32.
[80] G.Horváth, Á.:Relativity theory in time-space manifold. *Univ. J. of Physics and Appl.* **10/4**, (2016) 115–127.
[81] G.Horváth, Á., Maximal convex hull of connecting simplices, *Stud. Univ. Žilina Math. Ser.* **22** (2008), 7-19.
[82] G.Horváth, Á., On an extremal problem connected with simplices. *Beiträge zur Geometrie und Algebra* (2013),doi:10.1007/s13366-013-0151-9.
[83] G.Horváth, Á., Computations with Mathematica 10, an addendum to the paper "On the icosahedron inequality of László Fejes-Tóth". DOI: 10.13140/2.1.3416.6566 (2014) or http://www.math.bme.hu/~ghorvath/calculationmath.pdf
[84] G.Horváth, Á., Lángi, Zs., Maximum volume polytopes inscribed in the unit sphere. (2014) **181/2**, (2016) 341-354.
[85] G.Horváth, Á., Lángi, Zs., Spirova, M., Semi inner products and the concept of semi-polarity. *Results in Mathematics* DOI: 10.1007/s00025-015-0510-y (2015)
[86] Halmos, P. R.: *Measure Theory.* Springer-Verlag, New York, Heidelberg, Berlin, 1974.
[87] Haantjes, J.: Distance geometry. Curvature in abstract metric spaces, *Nederl. Akad. Wetensch, Proc.* **50** (1947), 496-508.
[88] Hazewinkel, M. ed.: *Normal distribution.* Encyclopedia of Mathematics, Springer, 2001.
[89] Heavens, A. *General Relativity* http://star-www.st-and.ac.uk/~hz4/gr/HeavensGR.pdf
[90] Hoffmann, L. M.: Measures on the space of convex bodies. *Adv. Geom.* **10** (2010), 477–486.
[91] Ikawa, T.: Euler-Savary's formula in Minkowski geometry, *Balkan Journal of Geometry* **8**2 (2003), 31-36.
[92] James, R.C.: Orthogonality in normed linear spaces. *Duke Math. J.* **12**, (1945) 291–301.
[93] James, R. C.: Orthogonality and linear functionals in normed linear spaces. *Bull. Amer. Math. Soc.* **53** (1947), 559–566.
[94] Klain D., Rota G-C.: *Introduction to Geometric Probability.* Cambridge Univ. Press, 1997.
[95] Knowles R. J., Cook, T. A.: *Some results on Auerbach bases for finite dimensional normed spaces.* Bull. Soc. Roy. Sci. Liège **42** (1973), 518–522.
[96] Knopp, K.: *Theory of Functions, Parts I and II, Two Volumes Bound as One, Part I.* New York: Dover, 1996.
[97] Koehler, D. O.: A note on some operator theory in certain semi inner-product spaces. *Proc. Amer. Math. Soc.* **30(2)** (1971) 363–366.
[98] Koehler, D., Rosenthal, P.: On isometries of normed linear spaces. *Studia Math.* **36** (1970), 215–218.
[99] Kolmogorov, A.N. *Basic concepts of probability theory.* (in russian) Izgatelctvo "Nauka", 1974.
[100] Landsberg, G.: Über die Krümmung in der Variationsrechnung, *Math. Ann.* **65** (1908), 313-349.
[101] Lángi, Zs.: On diagonalizable operators in Minkowski spaces with the Lipschitz property. *Linear Algebra and its Applications* **433/11–12** (2010) 2161-2167.
[102] Lenz, H.: *Eine Kennzeichnung des Ellipsoids.* Arch. Math. **8** (1957), 209–211.
[103] Ling, J. M.: On Dekster's angle measure in Minkowski spaces, *J. Geom.* **85** (2006), 72-76.
[104] Lippmann, H: Zur Winkeltheorie in zweidimensionalen Minkowski- und Finsler-Räumen, *Indagationes Mathematicae* **60** (1957), 162–170.
[105] Lippmann, H.: Metrische Eigenschaften verschiedener Winkelmaße im Minkowski- und Finslerraum, I, *Indagationes Mathematicae* **61** (1958), 223-230.
[106] Lippmann, H.: Metrische Eigenschaften verschiedener Winkelmaße im Minkowski- und Finslerraum, II, *Indagationes Mathematicae* **61** (1958), 231-238.
[107] Lee H., Lin D.: Haar Measure on Compact Groups. http://www.math.cuhk.edu.hk/course/math5012/Haar~

[108] Lumer, G.: Semi inner product spaces. *Trans. Amer. Math. Soc.* **100** (1961), 29-43.
[109] Lumer, G.: On the isometries of reflexive Orlicz spaces. *Ann. Inst. Fourier, Grenoble* **13** (1963) 99–109.
[110] Macbeath, A.M., *An extremal property of the hypersphere*, Proc. Cambridge Philos. Soc. **47** (1951), 245-247.
[111] Mann, H.: Untersuchungen über Wabenzellen bei allgemeiner Minkowskischer Metrik. *Mh. Math. Phys.* **42**, (1935) 417–424.
[112] Martini, H. and Mustafaev, Z., *Some applications of cross-section measures in Minkowski spaces*, Period. Math. Hungar. **53** (2006), 185-197.
[113] Martini, H., Spirova, M.: Reflections in strictly convex Minkowski planes. *Aequationes Math.* **78**, 71–85 (2009)
[114] Martini, H., Spirova, M., Strambach, K.: Geometric algebra of strictly convex Minkowski planes. *Aequationes Mathematicae* (2013) DOI: 10.1007/s00010-013-0204-z
[115] Martini, H., Swanepoel, K. J., Weiss, G.: *The geometry of Minkowski spaces - a survey. Part I.* Expositiones Mathematicae **19** (2001), 97–142.
[116] Martini, H., Swanepoel, K.: The geometry of Minkowski spaces - a survey. Part II. *Expositiones Mathematicae* **22(2)** (2004), 93–144.
[117] Martini, H., Swanepoel, K.J.: Antinorms and radon curves. *Aequationes Math.* **71**,(2006) 110–138.
[118] Martini, H., Wu, S.: Radial projections of bisectors in Minkowski spaces. *Extracta Mathematicae* **23(1)** (2008), 7–28.
[119] E. J. McShane: Linear functionals on certain Banach spaces. *Proc. Amer. Math. Soc.* **1** (1950), 402–408.
[120] Menger, K.: Untersuchungen über allgemeine Metrik. Vierte Untersuchung, *Math. Ann.* **103** (1930) 466–501.
[121] Miličic, P.: On the B-angle and g-angle in a normed space, *Journal of Inequalities in Pure and Applied Math.* **8/3** (2007), no. 4, Art. 99, 9p.
[122] Milnor, J.W., Hyperbolic geometry: The first 150 years. *Bull. Amer. Math. Soc. (N.S.)* **6/1** (1982), 9-24.
[123] Minkowski, H.,: *Raum und Zeit* Jahresberichte der Deutschen Mathematiker-Vereinigung, Leipzig, 1909.
[124] Mitchell, W.I.R, Repovs, D.: The topology of cell-like mappings. *Proceedings of the conference on Differential Geometry and Topology, Cala Gonone, 26–30 September 1988.* (1988) 265–300.
[125] Molchanov, I: *Theory of random sets* Springer 2005.
[126] Dawis, M.W., Moussong, G.: *Notes on Nonpositively Curved Polyhedra* Bolyai Society Mathematical Studies, 8, Budapest, 1999.
[127] N. Mutoh, *The polyhedra of maximal volume inscribed in the unit sphere and of minimal volume circumscribed about the unit sphere*, JCDCG, Lecture Notes in Computer Science **2866** (2002), 204–214.
[128] Pontriagin, L.S.: Smooth manifolds and its applications in homotopy theory. *Trudy. Mat. Inst. AN CCCP.* **45** (1955).
[129] Rattan, S. S.: *Theory of Machines*, Tata McGraw Hill, New Delhi, (2009).
[130] Rogers, C.A., Shephard G.C., *Convex bodies associated with a given convex body*, J. London Math. Soc. **33** (1958), 270-281.
[131] Rogers, C.A., Shephard G.C., *Some extremal problems for convex bodies*, Mathematika **5** (1958), 93-102.
[132] Rudin, W.: *Functional analysis.* McGraw-Hill Book Company 1973.
[133] Santaló, L.A.: *Integral Geometry and Geometric Probability.* Reading, MA: Addison-Wesley 1976.
[134] Schoenflies, A.: *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, II.* Leipzig, Teubner 1908.
[135] Schneider, R.: *Convex bodies: the Brunn-Minkowski theory,* Cambridge University Press, Cambridge, 1993.
[136] Stampfli, J. G.: Adjoint abelian operators on Banach space. *Canadian J. Math.* **21**, (1969), 505–512.
[137] Swingle, P.M.: An unnecessary condition in two theorems of analysis situs. *Bulletin of the Amer. Math. Soc.* **34** (1928), 607–618.
[138] Tamássy, L.: Finsler spaces corresponding to distance spaces. *Proc. of the Conf., Contemporary Geometry and Related Topics, Belgrade, Serbia and Montenegro, June 26–July 2* (2005), 485–495.
[139] Taylor, A. E.: *A geometric theorem and its application to biorthogonal systems.* Bull. Amer. Math. Soc. **53** (1947), 614–616.
[140] Thompson, A. C.: *Minkowski Geometry.* Cambridge University Press, Cambridge, 1996.
[141] Uspenskij, V.: A short proof of a theorem of Morton Brown on chains of cells. http://arXiv.math.GN/0404372v1
[142] Underhill, A. L.: Invariants of the function $F(x, y, x', y')$ in the calculus of variations, *Trans. Am. Math. Soc.* **9** (1908), 316–338.
[143] Weszely, T., *The mathematical works of J.Bolyai* (in Hungarian) Kriterion, Bukarest 1981.
[144] Wilansky, A.: *Functional Analysis.* Blaisdell, New York, 1964.
[145] Wilder, R.L.: *Topology of Manifolds.* Am. Math. Soc. Coll. Part V, XXXII, 1949.
[146] Wu, S., Ji, D., Alonso, J.: Metric ellipses in Minkowski planes. *Extracta Math.* **20** (2005), 273–280.
[147] Woods, A.C.: A characteristic property of ellipsoids. *Duke Math. J.* **36** (1969), 1–6.